

ON SINGER'S CONJECTURE FOR THE FIFTH ALGEBRAIC TRANSFER

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ABSTRACT. Let $P_k := \mathbb{F}_2[x_1, x_2, \dots, x_k]$ be the polynomial algebra in k variables with the degree of each x_i being 1, regarded as a module over the mod-2 Steenrod algebra \mathcal{A} , and let GL_k be the general linear group over the prime field \mathbb{F}_2 which acts naturally on P_k . We study the *hit problem*, set up by Frank Peterson, of finding a minimal set of generators for the polynomial algebra P_k as a module over the mod-2 Steenrod algebra, \mathcal{A} . These results are used to study the Singer algebraic transfer which is a homomorphism from the homology of the mod-2 Steenrod algebra, $\text{Tor}_{k,k+n}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$, to the subspace of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ consisting of all the GL_k -invariant classes of degree n .

In this paper, we explicitly compute the hit problem for $k = 5$ and the degree $7 \cdot 2^s - 5$ with s an arbitrary positive integer. Using this result, we show that Singer's conjecture for the algebraic transfer is true in the case $k = 5$ and the above degree.

1. INTRODUCTION

Let E^k be an elementary abelian 2-group of rank k . Then,

$$P_k := H^*(E^k) \cong \mathbb{F}_2[x_1, x_2, \dots, x_k],$$

a polynomial algebra in k variables x_1, x_2, \dots, x_k , each of degree 1. Here the cohomology is taken with coefficients in the prime field \mathbb{F}_2 of two elements.

Being the cohomology of a group, P_k is a module over the mod-2 Steenrod algebra, \mathcal{A} . The action of \mathcal{A} on P_k is determined by the elementary properties of the Steenrod squares Sq^i and the Cartan formula (see Steenrod and Epstein [22]).

A polynomial f in P_k is called *hit* if it can be written as a finite sum $f = \sum_{u \geq 0} Sq^{2^u}(h_u)$ for suitable polynomials h_u . That means f belongs to $\mathcal{A}^+ P_k$, where \mathcal{A}^+ denotes the augmentation ideal in \mathcal{A} .

Let GL_k be the general linear group over the field \mathbb{F}_2 . This group acts naturally on P_k by matrix substitution. Since the two actions of \mathcal{A} and GL_k upon P_k commute with each other, there is an action of GL_k on $QP_k := \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

Many authors study the *hit problem* of determination of a minimal set of generators for P_k as a module over the Steenrod algebra, or equivalently, a basis of QP_k . This problem has first been studied by Peterson [16], Wood [30], Singer [20], Priddy [18], who show its relationship to several classical problems in homotopy theory.

The vector space QP_k was explicitly calculated by Peterson [16] for $k = 1, 2$, by Kameko [12] for $k = 3$ and by Sum [23, 25] for $k = 4$. However, for $k > 4$, it is still open.

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For a nonnegative integer d , denote by $(P_k)_d$ the subspace of P_k consisting of all the homogeneous polynomials of degree d in P_k and by $(QP_k)_d$ the subspace of QP_k consisting of all the classes represented by the elements in $(P_k)_d$. In [20], Singer defined the algebraic transfer, which is a homomorphism

$$\varphi_k : \text{Tor}_{k,k+d}^A(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_k)_d^{GL_k}$$

from the homology of the Steenrod algebra to the subspace of $(QP_k)_d$ consisting of all the GL_k -invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $\text{Tor}_{k,k+d}^A(\mathbb{F}_2, \mathbb{F}_2)$. The hit problem and the algebraic transfer was studied by many authors (see Boardman [2], Bruner-Hà-Hưng [3], Janfada [7], Hà [8], Hưng [9, 10], Chơn-Hà [5, 6], Minami [14], Nam [15], Hưng-Quỳnh [11], Quỳnh [19], Sum-Tín [27], Sum [24], Tín-Sum [29] and others).

Singer showed in [20] that φ_k is an isomorphism for $k = 1, 2$. Boardman showed in [2] that φ_3 is also an isomorphism. However, for any $k \geq 4$, φ_k is not a monomorphism in infinitely many degrees (see Singer [20], Hưng [10].) Singer made the following conjecture.

Conjecture 1.1 (Singer [20]). *The algebraic transfer φ_k is an epimorphism for any $k \geq 0$.*

The conjecture is true for $k \leq 3$. Based on the results in [23, 25], it can be verified for $k = 4$.

The purpose of the paper is to verify this conjecture for $k = 5$ and the degree $7 \cdot 2^s - 5$. The following is the main result of the paper.

Theorem 1.2. *Singer's conjecture is true for $k = 5$ and the degree $7 \cdot 2^s - 5$ with s an arbitrary positive integer.*

To prove the theorem, we study the hit problem for $k = 5$ and the degree $7 \cdot 2^s - 5$. We have

Theorem 1.3. *Let $m = 7 \cdot 2^s - 5$ with s a positive integer. Then*

- i) $\dim(QP_5)_m = 191$ for $s = 1$, and $\dim(QP_5)_m = 1245$ for any $s \geq 2$.
- ii) $(QP_5)_m^{GL_5} = 0$ for any $s \geq 1$.

This theorem has been proved by Singer [20] for $s = 1$. In [10], Hưng computed the dimensions of $(QP_5)_m$ and $(QP_5)_m^{GL_5}$ for $s = 2$ by using computer calculation. However, the detailed proof was unpublished at the time of the writing.

The proof of Theorem 1.3 is long and very technical. One of our main tools is Kameko's homomorphism $\widetilde{Sq}_*^0 : QP_k \rightarrow QP_k$, which is induced by an \mathbb{F}_2 -linear map $\phi : P_k \rightarrow P_k$, given by

$$\phi(x) = \begin{cases} y, & \text{if } x = x_1 x_2 \dots x_k y^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_k$. The map ϕ is not an \mathcal{A} -homomorphism. However, $\phi Sq^{2i} = Sq^i \phi$ and $\phi Sq^{2i+1} = 0$ for any non-negative integer i .

For a positive integer n , by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1 \leq i \leq r} (2^{u_i} - 1)$, where $u_i > 0$.

Theorem 1.4 (see Kameko [12]). *Let d be a non-negative integer. If $\mu(2d+k) = k$, then*

$$(\widetilde{Sq}_*^0)_{(k,d)} := \widetilde{Sq}_*^0 : (QP_k)_{2d+k} \longrightarrow (QP_k)_d$$

is an isomorphism of GL_k -modules.

Denote by $\alpha(n)$ the number of ones in dyadic expansion of a positive integer n and by $\zeta(n)$ the greatest integer u such that n is divisible by 2^u . That means $n = 2^{\zeta(n)}m$ with m an odd integer. Set

$$t(k, d) = \max\{0, k - \alpha(d + k) - \zeta(d + k)\}.$$

Sum proved in [26] the following.

Theorem 1.5 (see Sum [26]). *Let d be an arbitrary non-negative integer. Then*

$$(\widetilde{Sq}_*)^0{}^{s-t} : (QP_k)_{k(2^s-1)+2^s d} \longrightarrow (QP_k)_{k(2^t-1)+2^t d}$$

is an isomorphism of GL_k -modules for every $s \geq t$ if and only if $t \geq t(k, d)$.

For $d = 2$, we have $t(5, 2) = 2$ and $5(2^s - 1) + 2^s d = 7 \cdot 2^s - 5$. So, by Theorem 1.5, $(\widetilde{Sq}_*)^0{}^{s-2} : (QP_5)_{7 \cdot 2^s - 5} \longrightarrow (QP_5)_{23}$ is an isomorphism of GL_5 -modules for every $s \geq 2$. Hence, we need only to prove Theorem 1.3 by computing $(QP_5)_{7 \cdot 2^s - 5}$ and $(QP_5)_{7 \cdot 2^s - 5}^{GL_5}$ for $s = 1, 2$.

From the results of Tangora [28], Lin [13] and Chen [4], we obtain

$$\mathrm{Tor}_{5, 7 \cdot 2^s}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle (Ph_1)^* \rangle, & \text{if } s = 1, \\ \langle (h_s g_{s-1})^* \rangle, & \text{if } s \geq 2, \end{cases}$$

and $h_s g_{s-1} \neq 0$, where h_s denote the Adams element in $\mathrm{Ext}_{\mathcal{A}}^{1, 2^s}(\mathbb{F}_2, \mathbb{F}_2)$, P is the Adams periodicity operator in [1] and $g_{s-1} \in \mathrm{Ext}_{\mathcal{A}}^{4, 2^{s+2}+2^{s+1}}(\mathbb{F}_2, \mathbb{F}_2)$ for $s \geq 2$. Hence, by Theorem 1.3(ii), the homomorphism

$$\varphi_5 : \mathrm{Tor}_{5, 7 \cdot 2^s}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (QP_5)_{7 \cdot 2^s - 5}^{GL_5}$$

is an epimorphism. Theorem 1.2 is proved.

Observe that in this case, φ_5 is not a monomorphism. So, our result confirms the one of Hưng.

Corollary 1.6 (See Hưng [10]). *There are infinitely many degrees in which φ_5 is not a monomorphism.*

This paper is organized as follows. In Section 2, we recall some needed information on the admissible monomials in P_k , Singer's criterion on the hit monomials and Kameko's homomorphism. Our results will be presented in Section 3. Finally, in Section 4, we list all the admissible monomials of degrees 10, 23 in P_5 .

Theorems 1.2, 1.3 and 1.5 have already been announced in [29].

2. PRELIMINARIES

In this section, we recall some needed information from Kameko [12] and Singer [21], which will be used in the next section.

Notation 2.1. We denote $\mathbb{N}_k = \{1, 2, \dots, k\}$ and

$$X_{\mathbb{J}} = X_{\{j_1, j_2, \dots, j_s\}} = \prod_{j \in \mathbb{N}_k \setminus \mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \dots, j_s\} \subset \mathbb{N}_k,$$

In particular, $X_{\mathbb{N}_k} = 1$, $X_{\emptyset} = x_1 x_2 \dots x_k$, $X_j = x_1 \dots \hat{x}_j \dots x_k$, $1 \leq j \leq k$, and $X := X_k \in P_{k-1}$.

Let $\alpha_i(a)$ denote the i -th coefficient in dyadic expansion of a non-negative integer a . That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \dots$, for $\alpha_i(a) = 0$ or 1 with $i \geq 0$.

Let $x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} \in P_k$. Denote $\nu_j(x) = a_j$, $1 \leq j \leq k$. Set

$$\mathbb{J}_t(x) = \{j \in \mathbb{N}_k : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geq 0$. Then, we have $x = \prod_{t \geq 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.2. For a monomial x in P_k , define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \dots, \omega_i(x), \dots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \dots, \nu_k(x)),$$

where $\omega_i(x) = \sum_{1 \leq j \leq k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)}$, $i \geq 1$. The sequence $\omega(x)$ is called the weight vector of x .

Let $\omega = (\omega_1, \omega_2, \dots, \omega_i, \dots)$ be a sequence of non-negative integers. The sequence ω is called the weight vector if $\omega_i = 0$ for $i \gg 0$.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

For a weight vector ω , we define $\deg \omega = \sum_{i \geq 0} 2^{i-1} \omega_i$. Denote by $P_k(\omega)$ the subspace of P_k spanned by all monomials y such that $\deg y = \deg \omega$, $\omega(y) \leq \omega$, and by $P_k^-(\omega)$ the subspace of P_k spanned by all monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$.

Definition 2.3. Let ω be a weight vector and f, g two polynomials of the same degree in P_k .

- i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+ P_k$. If $f \equiv 0$ then f is called *hit*.
- ii) $f \equiv_\omega g$ if and only if $f - g \in \mathcal{A}^+ P_k + P_k^-(\omega)$.

Obviously, the relations \equiv and \equiv_ω are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation \equiv_ω . Then, we have

$$QP_k(\omega) = P_k(\omega) / ((\mathcal{A}^+ P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

For a polynomial $f \in P_k$, we denote by $[f]$ the class in QP_k represented by f . If ω is a weight vector, then denote by $[f]_\omega$ the class represented by f . Denote by $|S|$ the cardinal of a set S .

It is easy to see that

$$QP_k(\omega) \cong QP_k^\omega := \langle \{[x] \in QP_k : x \text{ is admissible and } \omega(x) = \omega\} \rangle.$$

So, we get

$$(QP_k)_n = \bigoplus_{\deg \omega = n} QP_k^\omega \cong \bigoplus_{\deg \omega = n} QP_k(\omega).$$

Hence, we can identify the vector space $QP_k(\omega)$ with $QP_k^\omega \subset QP_k$.

For $1 \leq i \leq k$, define the \mathcal{A} -homomorphism $g_i : P_k \rightarrow P_k$, which is determined by $g_i(x_i) = x_{i+1}$, $g_i(x_{i+1}) = x_i$, $g_i(x_j) = x_j$ for $j \neq i, i+1$, $1 \leq i < k$, and $g_k(x_1) = x_1 + x_2$, $g_k(x_j) = x_j$ for $j > 1$. Note that the general linear group GL_k is generated by the matrices associated with g_i , $1 \leq i \leq k$, and the symmetric group Σ_k is generated by the ones associated with g_i , $1 \leq i < k$. So, a homogeneous polynomial $f \in P_k$ is an GL_k -invariant if and only if $g_i(f) \equiv f$ for $1 \leq i \leq k$. If $g_i(f) \equiv f$ for $1 \leq i < k$, then f is an Σ_k -invariant.

We note that the weight vector of a monomial is invariant under the permutation of the generators x_i , hence $QP_k(\omega)$ has an action of the symmetric group Σ_k . Furthermore, we have the following.

Lemma 2.4 (See Sum [26]). *Let ω be a weight vector. Then, $QP_k(\omega)$ is the GL_k -module.*

Definition 2.5. Let x, y be monomials of the same degree in P_k . We say that $x < y$ if and only if one of the following holds:

- i) $\omega(x) < \omega(y)$;
- ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

Definition 2.6. A monomial x is said to be inadmissible if there exist monomials y_1, y_2, \dots, y_m such that $y_t < x$ for $t = 1, 2, \dots, m$ and $x - \sum_{t=1}^m y_t \in \mathcal{A}^+ P_k$.

A monomial x is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in P_k is a minimal set of \mathcal{A} -generators for P_k in degree n .

Theorem 2.7 (See Kameko [12]). *Let x, y, w be monomials in P_k such that $\omega_i(x) = 0$ for $i > r > 0$, $\omega_s(w) \neq 0$ and $\omega_i(w) = 0$ for $i > s > 0$.*

- i) *If w is inadmissible, then xw^{2^r} is also inadmissible.*
- ii) *If w is strictly inadmissible, then wy^{2^s} is also strictly inadmissible.*

Now, we recall a result of Singer [21] on the hit monomials in P_k .

Definition 2.8. A monomial z in P_k is called a spike if $\nu_j(z) = 2^{t_j} - 1$ for t_j a non-negative integer and $j = 1, 2, \dots, k$. If z is a spike with $t_1 > t_2 > \dots > t_{r-1} \geq t_r > 0$ and $t_j = 0$ for $j > r$, then it is called the minimal spike.

In [21], Singer showed that if $\mu(n) \leq k$, then there exists uniquely a minimal spike of degree n in P_k .

Lemma 2.9 (See [17]). *All the spikes in P_k are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector ω is weakly decreasing and $\omega_1 \leq k$, then there is a spike z in P_k such that $\omega(z) = \omega$.*

The following is a criterion for the hit monomials in P_k .

Theorem 2.10 (See Singer [21]). *Suppose $x \in P_k$ is a monomial of degree n , where $\mu(n) \leq k$. Let z be the minimal spike of degree n . If $\omega(x) < \omega(z)$, then x is hit.*

This result implies a result of Wood, which original is a conjecture of Peterson [16].

Theorem 2.11 (See Wood [30]). *If $\mu(n) > k$, then $(QP_k)_n = 0$.*

Now, we recall some notations and definitions in [25], which will be used in the next sections. We set

$$\begin{aligned} P_k^0 &= \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} : a_1 a_2 \dots a_k = 0\} \rangle, \\ P_k^+ &= \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} : a_1 a_2 \dots a_k > 0\} \rangle. \end{aligned}$$

It is easy to see that P_k^0 and P_k^+ are the \mathcal{A} -submodules of P_k . Furthermore, we have the following.

Proposition 2.12. *We have a direct summand decomposition of the \mathbb{F}_2 -vector spaces $QP_k = QP_k^0 \oplus QP_k^+$. Here $QP_k^0 = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^0$ and $QP_k^+ = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+$.*

Definition 2.13. For any $1 \leq i < j \leq k$, we define the homomorphism $p_{(i;j)} : P_k \rightarrow P_{k-1}$ of algebras by substituting

$$p_{(i;j)}(x_u) = \begin{cases} x_u, & \text{if } 1 \leq u < i, \\ x_{j-1}, & \text{if } u = i, \\ x_{u-1}, & \text{if } i < u \leq k. \end{cases}$$

For $1 \leq i \leq k$, define the homomorphism $f_i : P_{k-1} \rightarrow P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leq j < i, \\ x_{j+1}, & \text{if } i \leq j < k. \end{cases}$$

Then, $p_{(i;j)}$ is a homomorphism of \mathcal{A} -modules and $p_{(i;j)}(f_i(y)) = y$ for any $y \in P_{k-1}$.

For a subset $B \subset P_k$, we denote $[B] = \{[f] : f \in B\}$. If $B \subset P_k(\omega)$, then we set $[B]_\omega = \{[f]_\omega : f \in B\}$. From Theorem 2.10, we see that if ω is the weight vector of a minimal spike in P_k , then $[B]_\omega = [B]$. Obviously, we have

Proposition 2.14. *It is easy to see that if B is a minimal set of generators for \mathcal{A} -module P_{k-1} in degree n , then $f(B) = \bigcup_{i=1}^k f_i(B)$ is a minimal set of generators for \mathcal{A} -module P_k^0 in degree n .*

From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k , $B_k^0(n) = B_k(n) \cap P_k^0$, $B_k^+(n) = B_k(n) \cap P_k^+$. For a weight vector ω of degree n , we set $B_k(\omega) = B_k(n) \cap P_k(\omega)$, $B_k^+(\omega) = B_k^+(n) \cap P_k(\omega)$.

Then, $[B_k(\omega)]_\omega$ and $[B_k^+(\omega)]_\omega$, are respectively the bases of the \mathbb{F}_2 -vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$.

For any monomials z, z_1, z_2, \dots, z_m in $P_k(\omega)$ with $m \geq 1$, we denote

$$\begin{aligned} \Sigma_k(z_1, z_2, \dots, z_m) &= \{\sigma z_t : \sigma \in \Sigma_k, 1 \leq t \leq m\} \subset P_k(\omega), \\ [B(z_1, z_2, \dots, z_m)]_\omega &= [B_k(\omega)]_\omega \cap \langle [\Sigma_k(z_1, z_2, \dots, z_m)]_\omega \rangle, \\ p(z) &= \sum_{y \in B_k(n) \cap \Sigma_k(z)} y. \end{aligned}$$

Obviously, $\langle [\Sigma_k(z_1, z_2, \dots, z_m)]_\omega \rangle$ is an Σ_k -submodule of $QP_k(\omega)$. Furthermore, it is the Σ_k -module generated by the set $\{[z_1]_\omega, [z_2]_\omega, \dots, [z_m]_\omega\}$.

3. PROOFS OF THE RESULTS

In this section we prove Theorem 1.3 by explicitly determining all admissible monomials of degree $7 \cdot 2^s - 5$ in P_5 . Using this results, we determine the space $(QP_5)_{7 \cdot 2^s - 5}^{GL_5}$ for all $s \geq 1$. Recall that by Theorem 1.5,

$$(\widetilde{Sq}_*^0)^{s-2} : (QP_5)_{7 \cdot 2^s - 5} \longrightarrow (QP_5)_{23}$$

is an isomorphism of GL_5 -modules for every $s \geq 2$. So, we need only to prove the theorem for $s = 1, 2$.

3.1. The case $s = 1$.

For $s = 1$, we have $7 \cdot 2^s - 5 = 9$. Since Kameko's homomorphism

$$(\widetilde{Sq}_*^0)_{(5,2)} : (QP_5)_9 \longrightarrow (QP_5)_2$$

is an epimorphism, we have $(QP_5)_9 \cong \text{Ker}(\widetilde{Sq}_*^0)_{(5,2)} \oplus (QP_5)_2$. It is easy to see that $B_5(2) = \{x_i x_j : 1 \leq i < j \leq 5\}$. So, we need only to compute $\text{Ker}(\widetilde{Sq}_*^0)_{(5,2)}$.

Lemma 3.1.1. *If x is an admissible monomial of degree 9 and $[x] \in \text{Ker}(\widetilde{Sq}_*^0)_{(5,2)}$, then either $\omega(x) = (3, 1, 1)$ or $\omega(x) = (3, 3)$.*

Proof. Observe that $z = x_1^7 x_2 x_3$ is the minimal spike of degree 9 in P_5 and $\omega(z) = (3, 1, 1)$. Since $[x] \neq 0$, by Theorem 2.10, either $\omega_1(x) = 3$ or $\omega_1(x) = 5$. If $\omega_1(x) = 5$, then $x = X_\emptyset y^2$ with y a monomial of degree 2 in P_5 . Since x is admissible, by Theorem 2.7, y is admissible. So, $(\widetilde{Sq}_*^0)_{(5,2)}([x]) = [y] \neq 0$. This contradicts the fact that $[x] \in \text{Ker}(\widetilde{Sq}_*^0)_{(5,2)}$, hence $\omega_1(x) = 3$. Then, we have $x = x_i x_j x_\ell y_1^2$ with y_1 an admissible monomial of degree 3 in P_5 . It is easy to see that either $\omega(y_1) = (1, 1)$ or $\omega(y_1) = (3, 0)$. The lemma is proved. \square

Using this lemma, we see that

$$\text{Ker}(\widetilde{Sq}_*^0)_{(5,2)} = (QP_5^0)_9 \bigoplus QP_5^+(3, 1, 1) \bigoplus QP_5^+(3, 3).$$

From a result in [24], we have $\dim(QP_5^0)_9 = 160$.

Proposition 3.1.2. *$B_5^+(3, 1, 1)$ is the set of the monomials $x_1 x_j^2 x_\ell^4 x_u x_v$ such that $(1, j, \ell, u, v)$ is a permutation of $(1, 2, 3, 4, 5)$, $j < \ell$ and $u < v$.*

Proof. Let x be an admissible monomial in P_5^+ and $\omega(x) = (3, 1, 1)$. Then $x = x_i x_j^2 x_\ell^4 x_u x_v$ with (i, j, ℓ, u, v) a permutation of $(1, 2, 3, 4, 5)$ and $i < u < v$. If $\ell < j$, then using the Cartan formula, we have

$$x = x_i x_j^4 x_\ell^2 x_u x_v + Sq^2(x_i x_j^2 x_\ell^2 x_u x_v) \pmod{(P_5^-(3, 1, 1))}.$$

So, x is inadmissible, hence $j < \ell$. If $j = 1$, then

$$x = x_i^2 x_j x_\ell^4 x_u x_v + x_i x_j x_\ell^4 x_u^2 x_v + x_i x_j x_\ell^4 x_u x_v^2 + Sq^1(x_i x_j x_\ell^4 x_u x_v).$$

Hence, x is inadmissible, so $1 < j$. Since $i < u < v$ and $j < \ell$, we obtain $i = 1$.

By a direct computation, we see that the set

$$\{[x_1 x_j^2 x_\ell^4 x_u x_v] : 1 < j < \ell, 1 < u < v\}$$

is linearly independent in QP_5 . The proposition follows. \square

By a similar argument as given in the proof of Proposition 3.1.2, we get the following.

Proposition 3.1.3. *$B_5^+(3, 3)$ is the set of the monomials $x_1 x_j^2 x_\ell^2 x_u x_v^3$ such that $(1, j, \ell, u, v)$ is a permutation of $(1, 2, 3, 4, 5)$, and $j < \ell$.*

It is easy to see that $|B_5(2)| = 10$, $|B_5^+(3, 1, 1)| = 6$ and $|B_5^+(3, 3)| = 15$. Hence, the first part of Theorem 1.3 is proved for $s = 1$.

By an easy computation, one gets

Lemma 3.1.4. $(QP_5)_2^{GL_5} = 0$.

Since Kameko's homomorphism is an epimorphism of GL_5 -modules, Lemma 3.1.4 implies $(QP_5)_9^{GL_5} = \text{Ker}(\widetilde{Sq}_*^0)_{(5,2)}^{GL_5}$.

Using the above results, we see that $\dim(\text{Ker}(\widetilde{Sq}_*^0)_{(5,2)}) = 181$ with the basis $\bigcup_{i=1}^6 [B_5(u_i)]$, where

$$\begin{aligned} u_1 &= x_1 x_2 x_3^7, \quad u_2 = x_1^3 x_2^3 x_3^3, \quad u_3 = x_1 x_2^3 x_3^5, \\ u_4 &= x_1 x_2 x_3^2 x_4^3 x_5^3, \quad u_5 = x_1 x_2 x_3^2 x_4^5, \quad u_6 = x_1 x_2^2 x_3^3 x_4^3. \end{aligned}$$

By a direct computation, we obtain the following lemma.

Lemma 3.1.5. *We have a direct summand decomposition of the Σ_5 -modules:*

$$(\text{Ker}(\widetilde{Sq}_*^0)_{(5,2)}) = \bigoplus_{i=1}^4 \langle [\Sigma_5(u_i)] \rangle \bigoplus \langle [\Sigma_5(u_5, u_6)] \rangle.$$

Lemma 3.1.6.

- i) $\langle [\Sigma_5(u_i)] \rangle^{\Sigma_5} = \langle [p(u_i)] \rangle$, $i = 1, 2, 3$.
- ii) $\langle [\Sigma_5(u_4)] \rangle^{\Sigma_5} = \langle [p_1], [p_2] \rangle$, where

$$\begin{aligned} p_1 &= x_1 x_2 x_3 x_4^2 x_5^4 + x_1 x_2 x_3^2 x_4 x_5^4 + x_1 x_2 x_3^2 x_4^4 x_5 \\ &\quad + x_1 x_2^2 x_3 x_4 x_5^4 + x_1 x_2^2 x_3 x_4^4 x_5 + x_1 x_2^2 x_3^4 x_4 x_5, \\ p_2 &= x_1 x_2 x_3 x_4^2 x_5^4 + x_1 x_2 x_3^2 x_4 x_5^4 + x_1 x_2 x_3^2 x_4^4 x_5 + x_1 x_2 x_3^2 x_4^2 x_5^3 + x_1 x_2 x_3^2 x_4^3 x_5^2 \\ &\quad + x_1 x_2 x_3^3 x_4^2 x_5^2 + x_1 x_2^2 x_3 x_4^2 x_5^3 + x_1 x_2^2 x_3 x_4^3 x_5^2 + x_1 x_2^2 x_3^2 x_4 x_5^3 + x_1 x_2^2 x_3^2 x_4^3 x_5^2 \\ &\quad + x_1 x_2^2 x_3^3 x_4 x_5^2 + x_1 x_2^2 x_3^3 x_4^2 x_5 + x_1 x_2^3 x_3 x_4^2 x_5^2 + x_1 x_2^3 x_3^2 x_4 x_5^2 + x_1 x_2^3 x_3^2 x_4^2 x_5 \\ &\quad + x_1^3 x_2 x_3 x_4^2 x_5^2 + x_1^3 x_2 x_3^2 x_4 x_5^2 + x_1^3 x_2 x_3^2 x_4^2 x_5. \end{aligned}$$

- iii) $\langle [\Sigma_5(u_5, u_6)] \rangle^{\Sigma_5} = \langle [p_3] \rangle$, where

$$\begin{aligned} p_3 &= x_2 x_3 x_4 x_5^6 + x_2 x_3 x_4^6 x_5 + x_2 x_3^6 x_4 x_5 + x_1 x_3 x_4 x_5^6 + x_1 x_3 x_4^6 x_5 + x_1 x_3^6 x_4 x_5 \\ &\quad + x_1 x_2 x_4 x_5^6 + x_1 x_2 x_4^6 x_5 + x_1 x_2 x_3 x_5^6 + x_1 x_2 x_3 x_4^6 + x_1 x_2 x_3^6 x_5 + x_1 x_2 x_3^6 x_4 \\ &\quad + x_1 x_2^6 x_4 x_5 + x_1 x_2^6 x_3 x_5 + x_1 x_2^6 x_3 x_4 + x_2^3 x_3 x_4 x_5^4 + x_2^3 x_3 x_4^4 x_5 + x_2^3 x_3^4 x_4 x_5 \\ &\quad + x_1^3 x_3 x_4 x_5^4 + x_1^3 x_3 x_4^4 x_5 + x_1^3 x_3^4 x_4 x_5 + x_1^3 x_2 x_4 x_5^4 + x_1^3 x_2 x_4^4 x_5 + x_1^3 x_2 x_3 x_5^4 \\ &\quad + x_1^3 x_2 x_3 x_4^4 + x_1^3 x_2 x_3^4 x_5 + x_1^3 x_2 x_3^4 x_4 + x_1^3 x_2^4 x_4 x_5 + x_1^3 x_2^4 x_3 x_5 + x_1^3 x_2^4 x_3 x_4. \end{aligned}$$

Proof. We prove the Part (ii) of the lemma. The others can be proved by a similar computation. From Propositions 3.1.2 and 3.1.3, we see that $\dim\langle [\Sigma_5(u_4)] \rangle = 21$ with a basis consisting of the classes represented by the following monomials:

$$\begin{aligned} v_1 &= x_1 x_2 x_3 x_4^2 x_5^4, \quad v_2 = x_1 x_2 x_3^2 x_4 x_5^4, \quad v_3 = x_1 x_2 x_3^2 x_4^4 x_5, \quad v_4 = x_1 x_2^2 x_3 x_4 x_5^4, \\ v_5 &= x_1 x_2^2 x_3 x_4^4 x_5, \quad v_6 = x_1 x_2^2 x_3^4 x_4 x_5, \quad v_7 = x_1 x_2 x_3^2 x_4^2 x_5^3, \quad v_8 = x_1 x_2 x_3^2 x_4^3 x_5^2, \\ v_9 &= x_1 x_2 x_3^3 x_4^2 x_5^2, \quad v_{10} = x_1 x_2^2 x_3 x_4^2 x_5^3, \quad v_{11} = x_1 x_2^2 x_3 x_4^3 x_5^2, \quad v_{12} = x_1 x_2^2 x_3^2 x_4 x_5^3, \\ v_{13} &= x_1 x_2^2 x_3^2 x_4^3 x_5^2, \quad v_{14} = x_1 x_2^2 x_3^3 x_4 x_5^2, \quad v_{15} = x_1 x_2^2 x_3^3 x_4^2 x_5, \quad v_{16} = x_1 x_2^3 x_3 x_4^2 x_5^2, \\ v_{17} &= x_1 x_2^3 x_3^2 x_4 x_5^2, \quad v_{18} = x_1 x_2^3 x_3^2 x_4^2 x_5, \quad v_{19} = x_1^3 x_2 x_3 x_4^2 x_5^2, \quad v_{20} = x_1^3 x_2 x_3^2 x_4 x_5^2, \\ v_{21} &= x_1^3 x_2 x_3^2 x_4^2 x_5. \end{aligned}$$

Suppose $f = \sum_{t=1}^{21} \gamma_t v_t$ with $\gamma_t \in \mathbb{F}_2$ and $[f] \in \langle [\Sigma_5(u_4)] \rangle^{\Sigma_5}$. By a direct computation, we have

$$\begin{aligned} g_1(f) + f &\equiv (\gamma_4 + \gamma_5 + \gamma_{10} + \gamma_{11})v_1 + (\gamma_4 + \gamma_6 + \gamma_{12} + \gamma_{14})v_2 \\ &\quad + (\gamma_5 + \gamma_6 + \gamma_{13} + \gamma_{15})v_3 + (\gamma_{10} + \gamma_{12})v_7 + (\gamma_{11} + \gamma_{13})v_8 \\ &\quad + (\gamma_{14} + \gamma_{15})v_9 + (\gamma_{16} + \gamma_{19})v_{16} + (\gamma_{17} + \gamma_{20})v_{17} + (\gamma_{18} + \gamma_{21})v_{18} \\ &\quad + (\gamma_{16} + \gamma_{19})v_{19} + (\gamma_{17} + \gamma_{20})v_{20} + (\gamma_{18} + \gamma_{21})v_{21} \equiv 0, \end{aligned}$$

This relation implies

$$\begin{cases} \gamma_4 + \gamma_5 + \gamma_{10} + \gamma_{11} = \gamma_4 + \gamma_6 + \gamma_{12} + \gamma_{14} = \gamma_5 + \gamma_6 + \gamma_{13} + \gamma_{15} = 0 \\ \gamma_{10} = \gamma_{12}, \gamma_{11} = \gamma_{13}, \gamma_{14} = \gamma_{15}, \gamma_{16} = \gamma_{19}, \gamma_{17} = \gamma_{20}, \gamma_{18} = \gamma_{21}. \end{cases} \quad (3.1)$$

With the aid of (3.1), we have

$$\begin{aligned} g_2(f) + f &\equiv (\gamma_{17} + \gamma_{18})v_1 + (\gamma_2 + \gamma_4 + \gamma_{17})v_2 + (\gamma_3 + \gamma_5 + \gamma_{18})v_3 \\ &\quad + (\gamma_2 + \gamma_4 + \gamma_{17})v_4 + (\gamma_3 + \gamma_5 + \gamma_{18})v_5 + (\gamma_7 + \gamma_{10})v_7 \\ &\quad + (\gamma_8 + \gamma_{11})v_8 + (\gamma_9 + \gamma_{16})v_9 + (\gamma_7 + \gamma_{10})v_{10} + (\gamma_8 + \gamma_{11})v_{11} \\ &\quad + (\gamma_{14} + \gamma_{17})v_{14} + (\gamma_{14} + \gamma_{18})v_{15} + (\gamma_9 + \gamma_{16})v_{16} \\ &\quad + (\gamma_{14} + \gamma_{17})v_{17} + (\gamma_{14} + \gamma_{18})v_{18} + (\gamma_{17} + \gamma_{18})v_{19} \equiv 0. \end{aligned}$$

From the last equality, we get

$$\begin{cases} \gamma_2 + \gamma_4 + \gamma_{14} = \gamma_3 + \gamma_5 + \gamma_{14} = 0, \\ \gamma_2 + \gamma_4 + \gamma_{14} = \gamma_3 + \gamma_5 + \gamma_{14} = 0, \\ \gamma_{14} = \gamma_{18}, \gamma_7 = \gamma_{10}, \gamma_8 = \gamma_{11}, \gamma_9 = \gamma_{16}, \gamma_{14} = \gamma_{17}. \end{cases} \quad (3.2)$$

By a direct computation using (3.1) and (3.2), we obtain

$$\begin{aligned} g_3(f) + f &\equiv (\gamma_1 + \gamma_2)v_1 + (\gamma_1 + \gamma_2)v_2 + (\gamma_5 + \gamma_6)v_5 + (\gamma_5 + \gamma_6)v_6 \\ &\quad + (\gamma_8 + \gamma_9)v_8 + (\gamma_8 + \gamma_9)v_9 + (\gamma_8 + \gamma_{14})v_{11} + (\gamma_8 + \gamma_{14})v_{13} \\ &\quad + (\gamma_8 + \gamma_{14})v_{14} + (\gamma_8 + \gamma_{14})v_{15} + (\gamma_9 + \gamma_{14})v_{16} \\ &\quad + (\gamma_9 + \gamma_{14})v_{17} + (\gamma_9 + \gamma_{14})v_{19} + (\gamma_9 + \gamma_{14})v_{20} \equiv 0. \end{aligned}$$

This implies

$$\gamma_1 + \gamma_2 = \gamma_5 + \gamma_6 = \gamma_8 + \gamma_9 = \gamma_8 + \gamma_{14} = 0. \quad (3.3)$$

By using (3.1), (3.2) and (3.3), we have

$$\begin{aligned} g_4(f) + f &\equiv (\gamma_1 + \gamma_3)(v_2 + v_3) + (\gamma_4 + \gamma_5)(v_4 + v_5) \\ &\quad + (\gamma_7 + \gamma_8)(v_7 + v_8 + v_{10} + v_{11} + v_{12} + v_{13}) \equiv 0. \end{aligned}$$

From this one gets

$$\gamma_1 + \gamma_3 = \gamma_4 + \gamma_5 = \gamma_7 + \gamma_8 = 0. \quad (3.4)$$

Part ii) of the lemma follows from (3.1)-(3.4). \square

Now, we prove the second part of Theorem 1.3 for $s = 1$.

Let $f \in (P_5)_9$ such that $[f] \in (QP_5)_9^{GL_5}$. Since $[f] \in (QP_5)_9^{\Sigma_5}$, using Lemmas 3.1.5 and 3.1.6, we have

$$f \equiv \gamma_1 p(u_1) + \gamma_2 p(u_2) + \gamma_3 p(u_3) + \gamma_4 p_1 + \gamma_5 p_2 + \gamma_6 p_3,$$

with $\gamma_j \in \mathbb{F}_2$. By computing $g_5(f) + f$ in terms of the admissible monomials, we obtain

$$\begin{aligned} g_5(f) + f \equiv & \gamma_1 x_2 x_4 x_5^7 + \gamma_2 x_2^3 x_3^3 x_4^3 + \gamma_3 x_2 x_4^3 x_5^5 + (\gamma_4 + \gamma_5 + \gamma_6) x_2^3 x_3 x_4 x_5^4 \\ & + \gamma_5 x_2 x_3 x_4^2 x_5^5 + (\gamma_5 + \gamma_6) x_2 x_3 x_4 x_5^6 + \text{other terms} \equiv 0. \end{aligned}$$

This relation implies $\gamma_j = 0$ for $1 \leq j \leq 6$. Theorem 1.3 is proved for $s = 1$.

3.2. The admissible monomials of degree 10 in P_5 .

To prove Theorem 1.3 for $s = 2$, we need to determine all the admissible monomials of degree 10 in P_5 .

Lemma 3.2.1. *If x is an admissible monomial of degree 10 in P_5 , then $\omega(x)$ is one of the following sequences: $(2, 2, 1)$, $(2, 4)$, $(4, 1, 1)$, $(4, 3)$.*

Proof. Observe that $z = x_1^7 x_2^3$ is the minimal spike of degree 10 in P_5 and $\omega(z) = (2, 2, 1)$. Since $[x] \neq 0$, by Theorem 2.10, either $\omega_1(x) = 2$ or $\omega_1(x) = 4$. If $\omega_1(x) = 2$, then $x = x_i x_j y^2$ with y a monomial of degree 4 in P_5 and $i < j$. Since x is admissible, by Theorem 2.7, y is admissible and $y \in P_5^0$. Using a result in [25], one gets either $\omega(y) = (2, 1)$ or $\omega(y) = (4, 0)$. If $\omega_1(x) = 4$, then $x = X_j y_1^2$ with y_1 a monomial of degree 3 in P_5 . Since y_1 is admissible, we see that either $\omega(y_1) = (1, 1)$ or $\omega(y_1) = (3, 0)$. The lemma is proved. \square

From this lemma and a result in [25], we have

$$\begin{aligned} (QP_5)_{10} = & (QP_5^0)_{10} \bigoplus QP_5^+(2, 2, 1) \\ & \bigoplus QP_5^+(2, 4) \bigoplus QP_5^+(4, 1, 1) \bigoplus QP_5^+(4, 3). \end{aligned}$$

Using a result in [25], we have $\dim(QP_5^0)_{10} = 230$.

Proposition 3.2.2.

- i) $B_5^+(2, 2, 1) = \{x_1 x_2 x_3^2 x_4^2 x_5^4, x_1 x_2 x_3^2 x_4^4 x_5^2, x_1 x_2^2 x_3 x_4^2 x_5^4, x_1 x_2^2 x_3 x_4^4 x_5^2, x_1 x_2^2 x_3^4 x_4 x_5^2\}$.
- ii) $B_5^+(2, 4) = \{x_1 x_2^2 x_3^2 x_4^2 x_5^3, x_1 x_2^2 x_3^2 x_4^4 x_5^2, x_1 x_2^2 x_3^3 x_4^2 x_5^2, x_1 x_2^2 x_3^2 x_4^2 x_5^5, x_1^3 x_2 x_3^2 x_4^2 x_5^2\}$.
- iii) $B_5^+(4, 1, 1)$ is the set of the following monomials:

$$\begin{aligned} & x_1 x_2 x_3 x_4 x_5^6, \quad x_1 x_2 x_3 x_4^6 x_5, \quad x_1 x_2 x_3^6 x_4 x_5, \quad x_1 x_2^6 x_3 x_4 x_5, \quad x_1 x_2 x_3 x_4^2 x_5^5, \\ & x_1 x_2 x_3^2 x_4 x_5^5, \quad x_1 x_2 x_3^2 x_4^5 x_5, \quad x_1 x_2^2 x_3 x_4 x_5^5, \quad x_1 x_2^2 x_3 x_4^5 x_5, \quad x_1 x_2^2 x_3^5 x_4 x_5, \\ & x_1 x_2 x_3 x_4^3 x_5^4, \quad x_1 x_2 x_3^3 x_4 x_5^4, \quad x_1 x_2 x_3^3 x_4^4 x_5, \quad x_1 x_2^3 x_3 x_4 x_5^4, \quad x_1 x_2^3 x_3 x_4^4 x_5, \\ & x_1 x_2^3 x_3^4 x_4 x_5, \quad x_1^3 x_2 x_3 x_4 x_5^4, \quad x_1^3 x_2 x_3 x_4^4 x_5, \quad x_1^3 x_2 x_3^4 x_4 x_5, \quad x_1^3 x_2^4 x_3 x_4 x_5. \end{aligned}$$

- iv) $B_5^+(4, 3)$ is the set of the following monomials:

$$\begin{aligned} & x_1 x_2 x_3^2 x_4^3 x_5^3, \quad x_1 x_2 x_3^3 x_4^2 x_5^3, \quad x_1 x_2 x_3^3 x_4^3 x_5^2, \quad x_1 x_2^2 x_3 x_4^3 x_5^3, \quad x_1 x_2^2 x_3^3 x_4 x_5^3, \\ & x_1 x_2^2 x_3^3 x_4^3 x_5, \quad x_1 x_2^3 x_3 x_4^2 x_5^3, \quad x_1 x_2^3 x_3 x_4^3 x_5^2, \quad x_1 x_2^3 x_3^2 x_4 x_5^3, \quad x_1 x_2^3 x_3^2 x_4^3 x_5, \\ & x_1 x_2^3 x_3^3 x_4 x_5^2, \quad x_1 x_2^3 x_3^3 x_4^2 x_5, \quad x_1^3 x_2 x_3 x_4^2 x_5^3, \quad x_1^3 x_2 x_3 x_4^3 x_5^2, \quad x_1^3 x_2 x_3^2 x_4 x_5^3, \\ & x_1^3 x_2 x_3^2 x_4^3 x_5, \quad x_1^3 x_2 x_3^3 x_4 x_5^2, \quad x_1^3 x_2 x_3^3 x_4^2 x_5, \quad x_1^3 x_2^2 x_3 x_4 x_5^2, \quad x_1^3 x_2^2 x_3 x_4^2 x_5. \end{aligned}$$

From the this proposition and a result in [25], we get $\dim(QP_5)_{10} = 280$. The proposition is proved by using Theorems 2.7, 2.10 and the following.

Lemma 3.2.3. *The following monomials are strictly inadmissible:*

- i) $x_j^2 x_\ell x_t^3$, $j < \ell$; $x_j^2 x_\ell x_t x_u^2$, $j < \ell < t$; $x_j x_\ell^2 x_t^2 x_u$, $j < \ell < t < u$; $x_1^2 x_2 x_3 x_4 x_5$.
- ii) $x_j^3 x_\ell^4 x_t^3$, $j < \ell < t$; $x_j^2 x_\ell^2 x_t^3 x_u^3$, $j < \ell < t$; $x_j^2 x_\ell x_t^2 x_u^2 x_v^3$, $j < \ell < t < u$;
 $x_j^2 x_\ell x_t x_u^3 x_v^3$, $j < \ell < t$.

Here (j, ℓ, t, u, v) is a permutation of $(1, 2, 3, 4, 5)$.

The proof of this lemma is straightforward.

Proof of Proposition 3.2.2. We prove the first part of the proposition. The others can be proved by a similar computation. We denote

$$\begin{aligned} a_1 &= x_1 x_2 x_3^2 x_4^2 x_5^4, a_2 = x_1 x_2 x_3^2 x_4^4 x_5^2, a_3 = x_1 x_2^2 x_3 x_4^2 x_5^4, \\ a_4 &= x_1 x_2^2 x_3 x_4^4 x_5^2, a_5 = x_1 x_2^2 x_3^4 x_4 x_5^2 \end{aligned}$$

Let x be an admissible monomial of degree 10 in P_5 such that $\omega(x) = (2, 2, 1)$. Then $x = x_i x_j y^2$ with $1 \leq i < j \leq 5$ and y a monomial of degree 4 in P_5 . Since x is admissible, according to Theorem 2.7, we have $y \in B_5(4)$.

By a direct computation we find that for all $y \in B_5(4)$, such that $x_i x_j y^2 \neq a_u$, $\forall u, 1 \leq u \leq 5$, there is a monomial w which is given in Lemma 3.2.3(i) such that $x_i x_j y^2 = w z^{2^r}$ with a monomial $z \in P_5$, and $r = \max\{t \in \mathbb{Z} : \omega_t(w) > 0\}$. By Theorem 2.7, $x_i x_j y^2$ is inadmissible. Since $x = x_i x_j y^2$ is admissible, one gets $x = a_u$ for suitable u .

We now prove the set $\{[a_u], 1 \leq u \leq 5\}$ is linearly independent in QP_5 . Suppose that $\mathcal{S} = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3 + \gamma_4 a_4 + \gamma_5 a_5 \equiv 0$ with $\gamma_u \in \mathbb{F}_2$. By a simple computation using Theorem 2.10, we have

$$\begin{aligned} p_{(1;2)}(\mathcal{S}) &\equiv \gamma_3 x_1^3 x_2 x_3^2 x_4^4 + \gamma_4 x_1^3 x_2 x_3^4 x_2 + \gamma_5 x_1^3 x_2^4 x_3 x_4^2 \equiv 0, \\ p_{(2;5)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_3) x_1 x_2 x_3^2 x_4^6 + \gamma_1 x_1 x_2^2 x_3 x_4^6 + \gamma_2 x_1 x_2^2 x_3^4 x_4^3 \equiv 0. \end{aligned}$$

These relations imply $\gamma_u = 0$ for all u . The first part of the proposition is proved. \square

3.3. The case $s = 2$.

For $s = 2$, we have $7 \cdot 2^s - 5 = 23$. Since Kameko's homomorphism

$$(\widetilde{Sq}_*)_{(5,9)}^0 : (QP_5)_{23} \longrightarrow (QP_5)_9$$

is an epimorphism, we have $(QP_5)_{23} \cong \text{Ker}(\widetilde{Sq}_*)_{(5,9)}^0 \oplus (QP_5)_9$. Hence, we need to compute $\text{Ker}(\widetilde{Sq}_*)_{(5,9)}^0$.

Lemma 3.3.1. *If x is an admissible monomial of degree 23 in P_5 and $[x] \in \text{Ker}(\widetilde{Sq}_*)_{(5,9)}^0$, then $\omega(x)$ is one of the following sequences:*

$$(3, 2, 2, 1), (3, 2, 4), (3, 4, 1, 1), (3, 4, 3).$$

Proof. Note that $z = x_1^{15} x_2^7 x_3$ is the minimal spike of degree 23 in P_5 and $\omega(z) = (3, 2, 2, 1)$. Since $[x] \neq 0$, by Theorem 2.10, either $\omega_1(x) = 3$ or $\omega_1(x) = 5$. If $\omega_1(x) = 5$, then $x = X_0 y^2$ with y a monomial of degree 9 in P_5 . Since x is admissible, by Theorem 2.7, y is admissible. Hence, $(\widetilde{Sq}_*)_{(5,9)}^0([x]) = [y] \neq 0$. This contradicts the fact that $[x] \in \text{Ker}(\widetilde{Sq}_*)_{(5,9)}^0$, so $\omega_1(x) = 3$. Then, we have $x = x_i x_j x_\ell y_1^2$ with y_1 an admissible monomial of degree 10 in P_5 . Now, the lemma follows from Lemma 3.2.1. \square

Using Lemma 3.3.1 and a result in [25], we get

$$\begin{aligned} \text{Ker}(\widetilde{Sq}_*)_{(5,9)}^0 &= (QP_5^0)_{23} \bigoplus (\text{Ker}(\widetilde{Sq}_*)_{(5,9)}^0 \cap (QP_5^+)_{23}), \\ \text{Ker}(\widetilde{Sq}_*)_{(5,9)}^0 \cap (QP_5^+)_{23} &= \bigoplus_{j=1}^4 QP_5^+(\omega_{(j)}). \end{aligned}$$

Here $\omega_{(1)} = (3, 2, 2, 1)$, $\omega_{(2)} = (3, 4, 1, 1)$, $\omega_{(3)} = (3, 4, 3)$, $\omega_{(4)} = (3, 2, 4)$. From a result in [25], we easily obtain $\dim(QP_5^0)_{23} = 635$. In this subsection, we prove the following.

Proposition 3.3.2. *The set $\{[b_t] : 1 \leq t \leq 419\}$ is a basis of the \mathbb{F}_2 -vector space $\text{Ker}(\widetilde{Sq}_*)_{(5,9)}^0 \cap (QP_5^+)_{23}$. Here the monomials $b_t = b_{23,t}$, with $1 \leq t \leq 419$, are determined as in Subsection 4.3.*

We prove this proposition by proving some lemmas.

Lemma 3.3.3. *The space $QP_5^+(\omega_{(1)})$ is spanned by the set $\{[a_t] : 1 \leq t \leq 290\}$.*

The following lemma is proved by a direct computation.

Lemma 3.3.4. *The following monomials are strictly inadmissible:*

i) $x_j^2 x_\ell x_t x_u^3$, $j < \ell < t$; $x_j^2 x_\ell x_t x_u^2$, $j < \ell < t < u$; $x_1 x_2^2 x_3^2 x_4 x_5$. Here (j, ℓ, t, u, v) is a permutation of $(1, 2, 3, 4, 5)$.

ii) $f_i(\bar{x})$, $1 \leq i \leq 5$, where \bar{x} is one of the following monomials:

$$\begin{aligned} &x_1^3 x_2^{12} x_3 x_4^7, \quad x_1^3 x_2^{12} x_3^7 x_4, \quad x_1^3 x_2^{12} x_3^3 x_4^5, \quad x_1^3 x_2^4 x_3^9 x_4^7, \\ &x_1^3 x_2^5 x_3^9 x_4^6, \quad x_1^3 x_2^5 x_3^8 x_4^7, \quad x_1^7 x_2^8 x_3^3 x_4^5. \end{aligned}$$

Lemma 3.3.5. *The following monomials are strictly inadmissible:*

$$\begin{aligned} &x_1 x_2^6 x_3^8 x_4^3 x_5^5, \quad x_1^3 x_2^4 x_3 x_4^8 x_5^7, \quad x_1^3 x_2^4 x_3 x_4^9 x_5^6, \quad x_1^3 x_2^4 x_3^3 x_4^4 x_5^9, \quad x_1^3 x_2^4 x_3^3 x_4^{12} x_5, \\ &x_1^3 x_2^4 x_3^8 x_4^8 x_5^7, \quad x_1^3 x_2^4 x_3^8 x_4^3 x_5^5, \quad x_1^3 x_2^4 x_3^8 x_4^7 x_5^5, \quad x_1^3 x_2^4 x_3^9 x_4^6 x_5^6, \quad x_1^3 x_2^4 x_3^9 x_4^6 x_5, \\ &x_1^3 x_2^4 x_3^{11} x_4^4 x_5, \quad x_1^3 x_2^5 x_3 x_4^8 x_5^6, \quad x_1^3 x_2^5 x_3^8 x_4^6 x_5^6, \quad x_1^3 x_2^5 x_3^8 x_4^6 x_5, \quad x_1^3 x_2^{12} x_3 x_4^6 x_5^6, \\ &x_1^3 x_2^{12} x_3 x_4^6 x_5, \quad x_1^3 x_2^{12} x_3^3 x_4^4 x_5, \quad x_1^7 x_2^8 x_3^3 x_4^4 x_5. \end{aligned}$$

Proof. We prove the lemma for the monomial $x = x_1 x_2^6 x_3^8 x_4^3 x_5^5$. The others can be proved by a similar computation. By a direct computation, we have

$$\begin{aligned} x &= x_1 x_2^4 x_3^{10} x_4^3 x_5^5 + x_1 x_2^4 x_3^6 x_4^3 x_5^9 + x_1 x_2^6 x_3^3 x_4^8 x_5^5 + x_1 x_2^8 x_3^3 x_4^6 x_5^5 + x_1 x_2^6 x_3^3 x_4^5 x_5^8 \\ &+ x_1 x_2^8 x_3^3 x_4^5 x_5^6 + x_1 x_2^6 x_3^3 x_4^8 x_5^5 + x_1 x_2^6 x_3^8 x_4^6 x_5^5 + x_1 x_2^3 x_3^8 x_4^5 x_5^6 + x_1 x_2^3 x_3^6 x_4^5 x_5^8 \\ &+ x_1 x_2^3 x_3^5 x_4^8 x_5^6 + x_1 x_2^3 x_3^5 x_4^6 x_5^8 + Sq^1(x_1^2 x_2^5 x_3^5 x_4^5 x_5^5) + Sq^2(x_1 x_2^6 x_3^6 x_4^3 x_5^5) \\ &+ x_1 x_2^5 x_3^5 x_4^5 x_5^5 + x_1 x_2^6 x_3^3 x_4^6 x_5^5 + x_1 x_2^6 x_3^3 x_4^5 x_5^6 + x_1 x_2^3 x_3^6 x_4^6 x_5^5 + x_1 x_2^3 x_3^6 x_4^5 x_5^6 \\ &+ x_1 x_2^3 x_3^5 x_4^6 x_5^6 + Sq^4(x_1 x_2^4 x_3^6 x_4^3 x_5^5) \pmod{(P_5^-(3, 2, 2, 1))}. \end{aligned}$$

Hence, the monomial x is strictly inadmissible. \square

Proof of Lemma 3.3.3. Let x be an admissible monomial in the space P_5^+ such that $\omega(x) = \omega_{(1)}$. Then $x = x_j x_\ell x_t y^2$ with $y \in B_5(2, 2, 1)$.

Let $z \in B_5(2, 2, 1)$ such that $x_j x_\ell x_t z^2 \in P_5^+$. By a direct computation using the results in Subsection 3.2, we see that if $x_j x_\ell x_t z^2 \neq b_t, \forall t, 1 \leq t \leq 290$, then there is a monomial w which is given in Lemma 3.3.4 such that $x_j x_\ell x_t z^2 = w z_1^{2^n}$ with suitable

monomial $z_1 \in P_5$, and $u = \max\{j \in \mathbb{Z} : \omega_j(w) > 0\}$. By Theorem 2.7, $x_j x_\ell x_t z^2$ is inadmissible. Since $x = x_j x_\ell x_t y^2$ with $y \in B_5(2, 2, 1)$ and x is admissible, one gets $x = b_t$ for suitable t . This implies $B_5^+(\omega_{(1)}) \subset \{b_t : 1 \leq t \leq 290\}$. The proposition follows. \square

Lemma 3.3.6. $B_5(\omega_{(4)}) = B_5^+(\omega_{(4)}) = \emptyset$. That means $QP_5(\omega_{(4)}) = 0$.

Proof. Let x be an admissible monomial in P_5^+ such that $\omega(x) = \omega_{(4)}$. Then $x = x_j x_\ell x_t y^2$ with $y \in B_5(2, 4)$. By a direct computation using Theorem 2.7, Proposition 3.2.2 and Lemma 3.3.4, we see that x is a permutation of one of the monomials: $x_1^3 x_2^4 x_3^4 x_4^5 x_5^7$, $x_1^3 x_2^4 x_3^5 x_4^5 x_5^6$. A simple computation shows:

$$\begin{aligned} x_1^3 x_2^4 x_3^4 x_4^5 x_5^7 &= Sq^1(x_1^3 x_2 x_3^2 x_4^9 x_5^7 + x_1^3 x_2 x_3^2 x_4^5 x_5^{11}) + Sq^2(x_1^5 x_2^2 x_3^2 x_4^5 x_5^7 + x_1^5 x_2 x_3^2 x_4^6 x_5^7) \\ &\quad + Sq^4(x_1^3 x_2^2 x_3^2 x_4^5 x_5^7 + x_1^3 x_2 x_3^2 x_4^6 x_5^7) \pmod{(P_5^-(B_5(\omega_{(4)})))}. \end{aligned}$$

This relation implies $[x_1^3 x_2^4 x_3^4 x_4^5 x_5^7]_{\omega_{(4)}} = 0$. By a similar computation, we have $[x_1^3 x_2^4 x_3^5 x_4^5 x_5^6]_{\omega_{(4)}} = 0$. The proposition is proved. \square

Lemma 3.3.7. The space $QP_5(\omega_{(2)})$ is spanned by the set $\{[b_t]_{\omega_{(2)}} : 291 \leq t \leq 395\}$, where the monomials b_t are determined as in Subsection 4.3.

Lemma 3.3.8. The following monomials are strictly inadmissible:

- i) $x_j^2 x_\ell x_t^2 x_u^3 x_v^3, i < j; x_j^2 x_\ell^3 x_t^3 x_u^3$.
Here (j, ℓ, t, u, v) is a permutation of $(1, 2, 3, 4, 5)$.
- ii) $x_1 x_2^2 x_3^6 x_4^3 x_5^3, x_1 x_2^6 x_3^2 x_4^3 x_5^3, x_1 x_2^6 x_3^3 x_4^2 x_5^3, x_1 x_2^6 x_3^3 x_4^3 x_5^2$.

The proof of this lemma is straightforward.

Proof of Lemma 3.3.7. Let x be an admissible monomial in P_5^+ such that $\omega(x) = \omega_{(2)}$. Then $x = x_j x_\ell x_t y^2$ with $y \in B_5(4, 1, 1)$.

Let $z \in B_5(4, 1, 1)$ such that $x_j x_\ell x_t z^2 \in P_5^+$. By a direct computation using Proposition 3.2.2, we see that if $x_j x_\ell x_t z^2 \notin b_t, \forall t, 291 \leq t \leq 395$, then there is a monomial w which is given in Lemma 3.3.8 such that $x_j x_\ell x_t z^2 = w z_1^{2^u}$ with suitable monomial $z_1 \in P_5$, and $u = \max\{j \in \mathbb{Z} : \omega_j(w) > 0\}$. By Theorem 2.7, $x_j x_\ell x_t z^2$ is inadmissible. Since $x = x_j x_\ell x_t y^2$ with $y \in B_5(4, 1, 1)$ and x is admissible, one gets $x = b_t$ for some t . The proposition is proved. \square

Lemma 3.3.9. The space $QP_5(\omega_{(3)})$ is spanned by the set $\{[a_t]_{\omega_{(3)}} : 396 \leq t \leq 419\}$, where the monomials b_t are determined as in Subsection 4.3.

The following lemma is proved by a direct computation.

Lemma 3.3.10. The following monomials are strictly inadmissible:

$$x_j x_\ell^6 x_t^3 x_u^6 x_v^7, j < \ell < t; x_j x_\ell^2 x_t^6 x_u^7 x_v^7.$$

Here (j, ℓ, t, u, v) is a permutation of $(1, 2, 3, 4, 5)$.

Proof of Lemma 3.3.9. Let x be an admissible monomial in P_5^+ such that $\omega(x) = \omega_{(3)}$. Then $x = x_j x_\ell x_t y^2$ with $y \in B_5(4, 3)$.

Let $z \in B_5(4, 3)$ such that $x_j x_\ell x_t z^2 \in P_5^+$. By a direct computation using Proposition 3.2.2, we see that if $x_j x_\ell x_t z^2 \neq b_t, \forall t, 396 \leq t \leq 419$, then there is a monomial w which is given in Lemma 3.3.10 such that $x_j x_\ell x_t z^2 = w z_1^{2^u}$ with suitable monomial $z_1 \in P_5$, and $u = \max\{j \in \mathbb{Z} : \omega_j(w) > 0\}$. By Theorem 2.7, $x_j x_\ell x_t z^2$ is inadmissible. Since $x = x_j x_\ell x_t y^2$ with $y \in B_5(4, 3)$ and x is admissible,

one gets $x = b_t$ for some t , $396 \leq t \leq 419$. This implies $B_5^+(\omega_{(3)}) \subset \{b_t : 396 \leq t \leq 419\}$. The proposition follows. \square

Proof of Proposition 3.3.2. From Lemmas 3.3.3, 3.3.6, 3.3.7 and 3.3.9 we see that the \mathbb{F}_2 -vector space $\text{Ker}(\widetilde{Sq}_*^0)_{(5,9)} \cap (QP_5^+)_{23}$ is spanned by the set $\{[b_t] : 1 \leq t \leq 419\}$. Now we prove that the set $\{[b_t] : 1 \leq t \leq 419\}$ is linearly independent in QP_5 . Suppose there is a linear relation

$$\mathcal{S} = \sum_{t=1}^{419} \gamma_t b_t \equiv 0,$$

where $\gamma_t \in \mathbb{F}_2$. We explicitly compute $p_{(i,j)}(\mathcal{S})$ in terms of the admissible monomials in P_4 . From the relations $p_{(i,j)}(\mathcal{S}) \equiv 0$ with $1 \leq i < j \leq 5$, one gets $\gamma_t = 0$ for all $1 \leq t \leq 419$. \square

Corollary 3.3.11. *Under the above notations, we have*

$$\dim QP_5(\omega_{(1)}) = 925, \dim QP_5(\omega_{(2)}) = 105, \dim QP_5(\omega_{(3)}) = 24.$$

Now we compute $(QP_5)_{23}^{GL_5}$. Since $(QP_5)_9^{GL_5} = 0$, using Theorem 2.7, we have $(QP_5)_{23}^{GL_5} = \text{Ker}(\widetilde{Sq}_*^0)_{(5,9)}^{GL_5}$. Recall that

$$\text{Ker}(\widetilde{Sq}_*^0)_{(5,9)} = QP_5(\omega_{(1)}) \bigoplus QP_5(\omega_{(2)}) \bigoplus QP_5(\omega_{(3)}).$$

By Lemma 3.3.9, $\dim QP_5(\omega_{(3)}) = 24$ with the basis $[B_5(\bar{b}_1)]_{\omega_{(3)}} \cup [B_5(\bar{b}_2)]_{\omega_{(3)}}$, where $\bar{b}_1 = x_1 x_2^3 x_3^6 x_4^6 x_5^7$, $\bar{b}_2 = x_1^3 x_2^3 x_3^5 x_4^6 x_5^6$.

Proposition 3.3.12. $QP_5(\omega_{(3)})^{GL_5} = 0$.

By a direct computation we easily obtain the following lemma.

Lemma 3.3.13. *We have a direct summand decomposition of the Σ_5 -modules:*

$$QP_5(\omega_{(3)}) = \langle [\Sigma_5(\bar{b}_1)]_{\omega_{(3)}} \rangle \bigoplus \langle [\Sigma_5(\bar{b}_2)]_{\omega_{(3)}} \rangle.$$

Lemma 3.3.14. $\langle [\Sigma_5(\bar{b}_1)]_{\omega_{(3)}} \rangle^{\Sigma_5} = \langle [p(\bar{b}_1)]_{\omega_{(3)}} \rangle$ and $\langle [\Sigma_5(\bar{b}_2)]_{\omega_{(3)}} \rangle^{\Sigma_5} = 0$.

Proof. From Lemma 3.3.9, we see that $\dim \langle [\Sigma_5(\bar{b}_2)]_{\omega_{(3)}} \rangle = 4$ with a basis consisting of the classes represented by the following monomials:

$$u_1 = x_1^3 x_2^3 x_3^5 x_4^6 x_5^6, \quad u_2 = x_1^3 x_2^5 x_3^3 x_4^6 x_5^6, \quad u_3 = x_1^3 x_2^5 x_3^6 x_4^3 x_5^6, \quad u_4 = x_1^3 x_2^5 x_3^6 x_4^6 x_5^3.$$

Suppose $f = \sum_{t=1}^4 \gamma_t u_t$ with $\gamma_t \in \mathbb{F}_2$ and $[f] \in \langle [\Sigma_5(\bar{b}_2)]_{\omega_{(3)}} \rangle^{\Sigma_5}$. By a direct computation, we have

$$\begin{aligned} g_1(f) + f &\equiv_{\omega_{(3)}} (\gamma_2 + \gamma_3 + \gamma_4) u_1 \equiv_{\omega_{(3)}} 0, \\ g_2(f) + f &\equiv_{\omega_{(3)}} (\gamma_1 + \gamma_2) u_1 \equiv_{\omega_{(3)}} 0, \\ g_3(f) + f &\equiv_{\omega_{(3)}} (\gamma_2 + \gamma_3) u_2 \equiv_{\omega_{(3)}} 0, \\ g_4(f) + f &\equiv_{\omega_{(3)}} (\gamma_3 + \gamma_4) u_3 \equiv_{\omega_{(3)}} 0. \end{aligned}$$

From the above relations one gets $\gamma_t = 0$ for $t = 1, 2, 3, 4$. By a similar computation we obtain $\langle [\Sigma_5(\bar{b}_1)]_{\omega_{(3)}} \rangle^{\Sigma_5} = \langle [p(\bar{b}_1)]_{\omega_{(3)}} \rangle$. \square

Proof of Proposition 3.3.12. Let $f \in P_5(\omega_{(3)})$ such that $[f]_{\omega_{(3)}} \in QP_5(\omega_{(3)})^{GL_5}$. Since $[f]_{\omega_{(3)}} \in QP_5(\omega_{(3)})^{\Sigma_5}$, using Lemmas 3.3.13, and 3.3.14, we have $f \equiv_{\omega_{(3)}} \gamma p(\bar{b}_1)$ with $\gamma \in \mathbb{F}_2$. By computing $g_5(f) + f$ in terms of the admissible monomials, we obtain

$$g_5(f) + f \equiv_{\omega_{(3)}} \gamma \bar{b}_1 + \text{other terms} \equiv_{\omega_{(3)}} 0.$$

This relation implies $\gamma = 0$. The proposition is proved. \square

Proposition 3.3.15. $QP_5(\omega_{(2)})^{GL_5} = 0$.

By computing from Lemma 3.3.7, we see that $\dim QP_5(\omega_{(2)}) = 105$ with the basis $\bigcup_{j=1}^4 [B(a_j)]_{\omega_{(2)}}$, where

$$a_1 = x_1 x_2^2 x_3^2 x_4^3 x_5^{15}, \quad a_2 = x_1 x_2^2 x_3^2 x_4^7 x_5^{11}, \quad a_3 = x_1 x_2^3 x_3^3 x_4^6 x_5^{10}, \quad a_4 = x_1 x_2^2 x_3^3 x_4^6 x_5^{11}.$$

By a direct computation, we obtain the following.

Lemma 3.3.16. *We have a direct summand decomposition of the Σ_5 -modules:*

$$QP_5(\omega_{(2)}) = \bigoplus_{j=1}^4 \langle [\Sigma_5(a_j)]_{\omega_{(2)}} \rangle.$$

Lemma 3.3.17. $\langle [\Sigma_5(a_j)]_{\omega_{(2)}} \rangle^{\Sigma_5} = \langle [p(a_j)]_{\omega_{(2)}} \rangle$, $j = 1, 2, 3$, and $\langle [\Sigma_5(a_4)]_{\omega_{(2)}} \rangle^{\Sigma_5} = \langle [p_4]_{\omega_{(2)}} \rangle$, where the polynomial p_4 is explicitly determined as in the Subsection 4.4.

Proof of Proposition 3.3.15. Let $f \in P_5(\omega_{(2)})$ such that $[f]_{\omega_{(2)}} \in QP_5(\omega_{(2)})^{GL_5}$. Since $[f]_{\omega_{(2)}} \in QP_5(\omega_{(2)})^{\Sigma_5}$, using Lemmas 3.3.16, and 3.3.17, we have

$$f \equiv_{\omega_{(2)}} \gamma_1 p(a_1) + \gamma_2 p(a_2) + \gamma_3 p(a_3) + \gamma_4 p_4,$$

with $\gamma_j \in \mathbb{F}_2$. By computing $g_5(f) + f$ in terms of the admissible monomials, we obtain

$$\begin{aligned} g_5(f) + f \equiv_{\omega_{(2)}} & \gamma_1 x_1^7 x_2^3 x_3^2 x_4^2 x_5^{10} + \gamma_2 x_1^3 x_2^3 x_3^2 x_4^2 x_5^{14} + \gamma_3 x_1 x_2^7 x_3^2 x_4^3 x_5^{10} \\ & + (\gamma_1 + \gamma_4) x_1^3 x_2^{13} x_3^2 x_4^2 x_5^3 + \text{other terms} \equiv_{\omega_{(2)}} 0. \end{aligned}$$

This relation implies $\gamma_j = 0$ with $j = 1, 2, 3, 4$. The proposition is proved. \square

Proposition 3.3.18. $QP_5(\omega_{(1)})^{GL_5} = 0$.

By using Proposition 3.3.6, we see that $\dim QP_5(\omega_{(1)}) = 925$. Consider the following monomials:

$$\begin{aligned} c_1 &= x_1 x_2^7 x_3^{15}, \quad c_2 = x_1^3 x_2^5 x_3^{15}, \quad c_3 = x_1^3 x_2^7 x_3^{13}, \quad c_4 = x_1 x_2^2 x_3^5 x_4^{15}, \quad c_5 = x_1 x_2 x_3^2 x_4^4 x_5^{15}, \\ c_6 &= x_1 x_2^2 x_3^7 x_4^{13}, \quad c_7 = x_1 x_2^3 x_3^5 x_4^{14}, \quad c_8 = x_1 x_2^3 x_3^6 x_4^{13}, \quad c_9 = x_1 x_2^3 x_3^7 x_4^{12}, \\ c_{10} &= x_1^3 x_2^5 x_3^6 x_4^9, \quad c_{11} = x_1 x_2 x_3^2 x_4^6 x_5^{13}, \quad c_{12} = x_1 x_2 x_3^2 x_4^7 x_5^{12}, \quad c_{13} = x_1 x_2 x_3^3 x_4^6 x_5^{12}, \\ c_{14} &= x_1 x_2^2 x_3^3 x_4^4 x_5^{13}, \quad c_{15} = x_1 x_2^2 x_3^3 x_4^5 x_5^{12}, \quad c_{16} = x_1 x_2^2 x_3^5 x_4^6 x_5^9, \quad c_{17} = x_1 x_2^2 x_3^5 x_4^7 x_5^8. \end{aligned}$$

Lemma 3.3.19. *We have a direct summand decomposition of the Σ_5 -modules:*

$$QP_5(\omega_{(1)}) = \bigoplus_{j=1}^5 \langle [\Sigma_5(a_j)] \rangle \bigoplus \langle [\Sigma_5(c_6, \dots, c_{10})] \rangle \bigoplus \langle [\Sigma_5(c_{11}, \dots, c_{17})] \rangle.$$

Lemma 3.3.20.

- i) $\langle [\Sigma_5(a_j)] \rangle^{\Sigma_5} = \langle [p(a_j)] \rangle$, $j = 1, 2, 3, 4, 5$.
- ii) $\langle [\Sigma_5(c_{11}, \dots, c_{17})] \rangle^{\Sigma_5} = 0$.
- iii) $\langle [\Sigma_5(c_6, \dots, c_{10})] \rangle^{\Sigma_5} = \langle [p_5 + p_6], [p_6 + p_7] \rangle$, where the polynomials p_5, p_6, p_7 are determined as in Subsection 4.4.

The proofs of the above lemmas are straightforward.

Proof of Proposition 3.3.18. Let $f \in P_5(\omega_{(1)})$ such that $[f] \in QP_5(\omega_{(1)})^{GL_5}$. Since $[f] \in QP_5(\omega_{(2)})^{\Sigma_5}$, using Lemmas 3.3.19, and 3.3.20, we have

$$f \equiv \sum_{j=1}^5 \gamma_j p(c_j) + \gamma_6(p_5 + p_6) + \gamma_7(p_6 + p_7),$$

with $\gamma_j \in \mathbb{F}_2$, $1 \leq j \leq 7$. By computing $g_5(f) + f$ in terms of the admissible monomials and using Theorem 2.10, we obtain

$$\begin{aligned} g_5(f) + f &\equiv \gamma_1 x_2 x_3^7 x_4^{15} + \gamma_2 x_2^3 x_3^{15} x_4^5 + \gamma_3 x_1^3 x_2^7 x_3^{13} + \gamma_4 x_2 x_3 x_4^6 x_5^{15} \\ &\quad + \gamma_5 x_1 x_2^{15} x_3 x_4^2 x_5^4 + \gamma_6 x_2 x_3 x_4^7 x_5^{14} + \gamma_7 x_2 x_3^3 x_4^5 x_5^{14} + \text{other terms} \equiv 0. \end{aligned}$$

This relation implies $\gamma_j = 0$ with $j = 1, 2, \dots, 7$. The proposition follows. \square

4. APPENDIX

In the appendix, we list all admissible monomials of degrees 9, 10, 23 in P_4 and P_5 . We order a set B of some monomials of degree n in P_k by using the order as in Definition 2.5.

4.1. The admissible monomials of degree 9 in P_5 .**4.1.1. The admissible monomials of degree 9 in P_4 .**

$B_4(9)$ is the set of 46 monomials $a_t = a_{9,t}$, $1 \leq t \leq 46$:

- | | | | | |
|---------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 1. $x_2 x_3 x_4^7$ | 2. $x_2 x_3^3 x_4^5$ | 3. $x_2 x_3^7 x_4$ | 4. $x_2^3 x_3 x_4^5$ | 5. $x_2^3 x_3^5 x_4$ |
| 6. $x_2^7 x_3 x_4$ | 7. $x_1 x_3 x_4^7$ | 8. $x_1 x_3^3 x_4^5$ | 9. $x_1 x_3^7 x_4$ | 10. $x_1 x_2 x_4^7$ |
| 11. $x_1 x_2 x_3 x_4^6$ | 12. $x_1 x_2 x_3^2 x_4^5$ | 13. $x_1 x_2 x_3^3 x_4^4$ | 14. $x_1 x_2 x_3^6 x_4$ | 15. $x_1 x_2 x_3^7$ |
| 16. $x_1 x_2^2 x_3 x_4^5$ | 17. $x_1 x_2^2 x_3^5 x_4$ | 18. $x_1 x_2^3 x_4^5$ | 19. $x_1 x_2^3 x_3 x_4^4$ | 20. $x_1 x_2^3 x_3^4 x_4$ |
| 21. $x_1 x_2^3 x_3^5$ | 22. $x_1 x_2^6 x_3 x_4$ | 23. $x_1 x_2^7 x_4$ | 24. $x_1 x_2^7 x_3$ | 25. $x_1^3 x_3 x_4^5$ |
| 26. $x_1^3 x_3^5 x_4$ | 27. $x_1^3 x_2 x_4^5$ | 28. $x_1^3 x_2 x_3 x_4^4$ | 29. $x_1^3 x_2 x_3^4 x_4$ | 30. $x_1^3 x_2 x_3^5$ |
| 31. $x_1^3 x_2^4 x_3 x_4$ | 32. $x_1^3 x_2^5 x_4$ | 33. $x_1^3 x_2^5 x_3$ | 34. $x_1^7 x_3 x_4$ | 35. $x_1^7 x_2 x_4$ |
| 36. $x_1^7 x_2 x_3$ | 37. $x_2^3 x_3^3 x_4^3$ | 38. $x_1 x_2^2 x_3^3 x_4^3$ | 39. $x_1 x_2^3 x_3^2 x_4^3$ | 40. $x_1 x_2^3 x_3^3 x_4^2$ |
| 41. $x_1^3 x_3^3 x_4^3$ | 42. $x_1^3 x_2 x_3^2 x_4^3$ | 43. $x_1^3 x_2 x_3^3 x_4^2$ | 44. $x_1^3 x_3^3 x_4^2$ | 45. $x_1^3 x_2^3 x_3 x_4^2$ |
| 46. $x_1^3 x_2^3 x_3^3$ | | | | |

4.1.2. The admissible monomials of degree 9 in P_5 .

$B_5(9) = B_5^0(9) \cup B_5^+(9)$, where $B_5^0(9) = \Phi^0(B_4(9))$, $|B_5^0(9)| = 160$ and $B_5^+(9) = B_5^+(3, 1, 1) \cup B_5^+(3, 3) \cup B_5(5, 2)$, where.

$B_5^+(3, 1, 1)$ is the set of 6 monomials $b_t = b_{9,t}$, $1 \leq t \leq 6$:

- | | | | |
|------------------------------|------------------------------|------------------------------|------------------------------|
| 1. $x_1 x_2 x_3 x_4^2 x_5^4$ | 2. $x_1 x_2 x_3^2 x_4 x_5^4$ | 3. $x_1 x_2 x_3^2 x_4^4 x_5$ | 4. $x_1 x_2^2 x_3 x_4 x_5^4$ |
| 5. $x_1 x_2^2 x_3 x_4^4 x_5$ | 6. $x_1 x_2^2 x_3^4 x_4 x_5$ | | |

$B_5^+(3, 3)$ is the set of 15 monomials $b_t = b_{9,t}$, $7 \leq t \leq 21$:

7. $x_1x_2x_3^2x_4^3x_5^3$ 8. $x_1x_2x_3^2x_4^3x_5^2$ 9. $x_1x_2x_3^3x_4^2x_5^2$ 10. $x_1x_2^2x_3x_4^2x_5^3$
 11. $x_1x_2^2x_3x_4^3x_5^2$ 12. $x_1x_2^2x_3^2x_4x_5^3$ 13. $x_1x_2^2x_3^2x_4^3x_5$ 14. $x_1x_2^2x_3^3x_4x_5^2$
 15. $x_1x_2^2x_3^3x_4^2x_5$ 16. $x_1x_2^3x_3x_4^2x_5^2$ 17. $x_1x_2^3x_3^2x_4x_5^2$ 18. $x_1x_2^3x_3^2x_4^2x_5$
 19. $x_1^3x_2x_3x_4^2x_5^2$ 20. $x_1^3x_2x_3^2x_4x_5^2$ 21. $x_1^3x_2x_3^2x_4^2x_5$.

$B_5(5, 2)$ is the set of 10 monomials $b_t = b_{9,t}$, $22 \leq t \leq 31$:

22. $x_1x_2x_3x_4^3x_5^3$ 23. $x_1x_2x_3^3x_4x_5^3$ 24. $x_1x_2x_3^3x_4^3x_5$ 25. $x_1x_2^3x_3x_4x_5^3$
 26. $x_1x_2^3x_3x_4^3x_5$ 27. $x_1x_2^3x_3^3x_4x_5$ 28. $x_1^3x_2x_3x_4x_5^3$ 29. $x_1^3x_2x_3x_4^3x_5$
 30. $x_1^3x_2x_3^3x_4x_5$ 31. $x_1^3x_2^3x_3x_4x_5$.

4.2. The admissible monomials of degree 10 in P_5 .

4.2.1. The admissible monomials of degree 10 in P_4 .

$B_4(10)$ is the set of 70 monomials.

1. $x_3^3x_4^7$ 2. $x_3^7x_4^3$ 3. $x_2x_3^2x_4^7$ 4. $x_2x_3^3x_4^6$ 5. $x_2x_3^6x_4^3$
 6. $x_2x_3^7x_4^2$ 7. $x_2^3x_4^7$ 8. $x_2^3x_3x_4^6$ 9. $x_2^3x_3^3x_4^4$ 10. $x_2^3x_3^5x_4^2$
 11. $x_2^3x_3^7$ 12. $x_2^7x_3^4$ 13. $x_2^7x_3x_4^2$ 14. $x_2^7x_3^3$ 15. $x_1x_2^2x_4^7$
 16. $x_1x_2^3x_4^6$ 17. $x_1x_2^6x_3^4$ 18. $x_1x_2^7x_3^2$ 19. $x_1x_2x_3^2x_4^6$ 20. $x_1x_2x_3^6x_4^2$
 21. $x_1x_2^2x_4^7$ 22. $x_1x_2^2x_3x_4^6$ 23. $x_1x_2^2x_3^3x_4^4$ 24. $x_1x_2^2x_3^4x_4^3$ 25. $x_1x_2^2x_3^5x_4^2$
 26. $x_1x_2^2x_3^7$ 27. $x_1x_2^3x_4^6$ 28. $x_1x_2^3x_3^2x_4^4$ 29. $x_1x_2^3x_3^4x_4^2$ 30. $x_1x_2^3x_3^6$
 31. $x_1x_2^6x_3^4$ 32. $x_1x_2^6x_3x_4^2$ 33. $x_1x_2^6x_3^3$ 34. $x_1x_2^7x_4^2$ 35. $x_1x_2^7x_3^2$
 36. $x_1^3x_4^7$ 37. $x_1^3x_3x_4^6$ 38. $x_1^3x_3^3x_4^4$ 39. $x_1^3x_3^5x_4^2$ 40. $x_1^3x_3^7$
 41. $x_1^3x_2x_4^6$ 42. $x_1^3x_2x_3^2x_4^4$ 43. $x_1^3x_2x_3^4x_4^2$ 44. $x_1^3x_2x_3^6$ 45. $x_1^3x_2^3x_4^4$
 46. $x_1^3x_2^3x_4^3$ 47. $x_1^3x_2^4x_3x_4^2$ 48. $x_1^3x_2^5x_4^2$ 49. $x_1^3x_2^5x_3^2$ 50. $x_1^3x_2^7$
 51. $x_1^7x_4^3$ 52. $x_1^7x_3x_4^2$ 53. $x_1^7x_3^3$ 54. $x_1^7x_2x_4^2$ 55. $x_1^7x_2x_3^2$
 56. $x_1^7x_2^3$ 57. $x_1x_2x_3x_4^7$ 58. $x_1x_2x_3^3x_4^5$ 59. $x_1x_2x_3^7x_4$ 60. $x_1x_2^3x_3x_4^5$
 61. $x_1x_2^3x_3^5x_4$ 62. $x_1x_2^7x_3x_4$ 63. $x_1^3x_2x_3x_4^5$ 64. $x_1^3x_2x_3^5x_4$ 65. $x_1^3x_2^5x_3x_4$
 66. $x_1^7x_2x_3x_4$ 67. $x_1x_2^3x_3^3x_4^3$ 68. $x_1^3x_2x_3^3x_4^3$ 69. $x_1^3x_2^3x_3x_4^3$ 70. $x_1^3x_2^3x_3^3x_4$

4.2.2. The admissible monomials of degree 10 in P_5 .

$B_5(10) = B_5^0(10) \cup B_5^+(10)$, where $B_5^0(10) = \Phi^0(B_4(10))$, $|B_5^0(10)| = 230$ and

$$B_5^+(10) = B_5^+(2, 2, 1) \cup B_5^+(2, 4) \cup B_5(4, 1, 1) \cup B_5(4, 3).$$

$B_5^+(2, 2, 1)$ is the set of 5 monomials:

$$x_1x_2x_3^2x_4^4x_5^4 \quad x_1x_2x_3^2x_4^4x_5^2 \quad x_1x_2^2x_3x_4^4x_5^4 \quad x_1x_2^2x_3x_4^4x_5^2 \quad x_1x_2^2x_3^4x_4x_5^2.$$

$B_5^+(2, 4)$ is the set of 5 monomials:

$$x_1x_2^2x_3^2x_4^3x_5^3 \quad x_1x_2^2x_3^2x_4^3x_5^2 \quad x_1x_2^2x_3^3x_4^2x_5^2 \quad x_1x_2^3x_3^2x_4^2x_5^2 \quad x_1^3x_2x_3^2x_4^2x_5^2.$$

$B_5^+(4, 1, 1)$ is the set of 20 monomials:

1. $x_1x_2x_3x_4x_5^6$ 2. $x_1x_2x_3x_4^2x_5^5$ 3. $x_1x_2x_3x_4^3x_5^4$ 4. $x_1x_2x_3x_4^6x_5$
 5. $x_1x_2x_3^2x_4x_5^5$ 6. $x_1x_2x_3^2x_4^5x_5$ 7. $x_1x_2x_3^3x_4x_5^4$ 8. $x_1x_2x_3^3x_4^4x_5$
 9. $x_1x_2x_3^6x_4x_5$ 10. $x_1x_2^2x_3x_4x_5^5$ 11. $x_1x_2^2x_3x_4^5x_5$ 12. $x_1x_2^2x_3^5x_4x_5$
 13. $x_1x_2^3x_3x_4x_5^4$ 14. $x_1x_2^3x_3x_4^4x_5$ 15. $x_1x_2^3x_3^4x_4x_5$ 16. $x_1x_2^6x_3x_4x_5$
 17. $x_1^3x_2x_3x_4x_5^4$ 18. $x_1^3x_2x_3x_4^4x_5$ 19. $x_1^3x_2x_3^4x_4x_5$ 20. $x_1^3x_2^4x_3x_4x_5$

$B_5^+(4, 3)$ is the set of 20 monomials:

- | | | | |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 1. $x_1x_2x_3^2x_4^3x_5^3$ | 2. $x_1x_2x_3^3x_4^2x_5^3$ | 3. $x_1x_2x_3^3x_4^3x_5^2$ | 4. $x_1x_2^2x_3x_4^3x_5^3$ |
| 5. $x_1x_2^2x_3^3x_4x_5^3$ | 6. $x_1x_2^2x_3^3x_4^3x_5$ | 7. $x_1x_2^2x_3x_4^2x_5^3$ | 8. $x_1x_2^2x_3x_4^3x_5^2$ |
| 9. $x_1x_2^3x_3^2x_4x_5^3$ | 10. $x_1x_2^3x_3^2x_4^3x_5$ | 11. $x_1x_2^3x_3^3x_4x_5^2$ | 12. $x_1x_2^3x_3^3x_4^2x_5$ |
| 13. $x_1^3x_2x_3x_4^2x_5^3$ | 14. $x_1^3x_2x_3x_4^3x_5^2$ | 15. $x_1^3x_2x_3^2x_4x_5^3$ | 16. $x_1^3x_2x_3^2x_4^3x_5$ |
| 17. $x_1^3x_2x_3^3x_4x_5^2$ | 18. $x_1^3x_2x_3^3x_4^2x_5$ | 19. $x_1^3x_2^3x_3x_4x_5^2$ | 20. $x_1^3x_2^3x_3x_4^2x_5$ |

4.3. The admissible monomials of degree 23 in P_5 .

4.3.1. The admissible monomials of degree 23 in P_4 .

$B_4(23)$ is the set of 155 monomials $a_t = a_{23,t}$, $1 \leq t \leq 155$:

- | | | | |
|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1. $x_1x_2x_3^7x_4^{14}$ | 2. $x_1x_2x_3^{14}x_4^7$ | 3. $x_1x_2^2x_3^7x_4^{13}$ | 4. $x_1x_2^2x_3^{13}x_4^7$ |
| 5. $x_1x_2^3x_3^5x_4^{14}$ | 6. $x_1x_2^3x_3^6x_4^{13}$ | 7. $x_1x_2^3x_3^7x_4^{12}$ | 8. $x_1x_2^3x_3^{12}x_4^7$ |
| 9. $x_1x_2^3x_3^{13}x_4^6$ | 10. $x_1x_2^3x_3^{14}x_4^5$ | 11. $x_1x_2^6x_3^3x_4^{13}$ | 12. $x_1x_2^6x_3^7x_4^9$ |
| 13. $x_1x_2^6x_3^{11}x_4^5$ | 14. $x_1x_2^7x_3^3x_4^{14}$ | 15. $x_1x_2^7x_3^2x_4^{13}$ | 16. $x_1x_2^7x_3^3x_4^{12}$ |
| 17. $x_1x_2^7x_3^6x_4^9$ | 18. $x_1x_2^7x_3^7x_4^8$ | 19. $x_1x_2^7x_3^{10}x_4^5$ | 20. $x_1x_2^7x_3^{11}x_4^4$ |
| 21. $x_1x_2^7x_3^{14}x_4$ | 22. $x_1x_2^{14}x_3^3x_4^7$ | 23. $x_1x_2^{14}x_3^3x_4^5$ | 24. $x_1x_2^{14}x_3^7x_4$ |
| 25. $x_1^3x_2x_3^5x_4^{14}$ | 26. $x_1^3x_2x_3^6x_4^{13}$ | 27. $x_1^3x_2x_3^7x_4^{12}$ | 28. $x_1^3x_2x_3^{12}x_4^7$ |
| 29. $x_1^3x_2x_3^{13}x_4^6$ | 30. $x_1^3x_2x_3^{14}x_4^5$ | 31. $x_1^3x_3^3x_4^{13}$ | 32. $x_1^3x_3^5x_4^{12}$ |
| 33. $x_1^3x_2^3x_3^{12}x_4^5$ | 34. $x_1^3x_2^3x_3^{13}x_4^4$ | 35. $x_1^3x_2^4x_3^3x_4^{13}$ | 36. $x_1^3x_2^4x_3^7x_4^9$ |
| 37. $x_1^3x_2^4x_3^{11}x_4^5$ | 38. $x_1^3x_2^5x_3^3x_4^{14}$ | 39. $x_1^3x_2^5x_3^2x_4^{13}$ | 40. $x_1^3x_2^5x_3^3x_4^{12}$ |
| 41. $x_1^3x_2^5x_3^6x_4^9$ | 42. $x_1^3x_2^5x_3^7x_4^8$ | 43. $x_1^3x_2^5x_3^{10}x_4^5$ | 44. $x_1^3x_2^5x_3^{11}x_4^4$ |
| 45. $x_1^3x_2^5x_3^{14}x_4$ | 46. $x_1^3x_2^7x_3^3x_4^{12}$ | 47. $x_1^3x_2^7x_3^4x_4^9$ | 48. $x_1^3x_2^7x_3^5x_4^8$ |
| 49. $x_1^3x_2^7x_3^8x_4^4$ | 50. $x_1^3x_2^7x_3^9x_4^4$ | 51. $x_1^3x_2^7x_3^{12}x_4$ | 52. $x_1^3x_2^{13}x_3x_4^6$ |
| 53. $x_1^3x_2^{13}x_3^2x_4^5$ | 54. $x_1^3x_2^{13}x_3^3x_4^4$ | 55. $x_1^3x_2^{13}x_3^6x_4$ | 56. $x_1^7x_2x_3x_4^{14}$ |
| 57. $x_1^7x_2x_3^2x_4^{13}$ | 58. $x_1^7x_2x_3^3x_4^{12}$ | 59. $x_1^7x_2x_3^6x_4^9$ | 60. $x_1^7x_2x_3^7x_4^8$ |
| 61. $x_1^7x_2x_3^{10}x_4^5$ | 62. $x_1^7x_2x_3^{11}x_4^4$ | 63. $x_1^7x_2x_3^{14}x_4$ | 64. $x_1^7x_3^3x_3x_4^{12}$ |
| 65. $x_1^7x_2^3x_3^4x_4^9$ | 66. $x_1^7x_2^3x_3^5x_4^8$ | 67. $x_1^7x_2^3x_3^8x_4^5$ | 68. $x_1^7x_2^3x_3^9x_4^4$ |
| 69. $x_1^7x_2^3x_3^{12}x_4$ | 70. $x_1^7x_2^3x_3^8x_4^4$ | 71. $x_1^7x_2^7x_3^8x_4$ | 72. $x_1^7x_2^9x_3^2x_4^5$ |
| 73. $x_1^7x_2^9x_3^4x_4$ | 74. $x_1^7x_2^{11}x_3^3x_4^4$ | 75. $x_1^7x_2^{11}x_3^4x_4$ | 76. $x_2x_7x_4^{15}$ |
| 77. $x_2x_3^{15}x_4^7$ | 78. $x_2^3x_3^5x_4^{15}$ | 79. $x_2^3x_7x_4^{13}$ | 80. $x_2^3x_3^{13}x_4^7$ |
| 81. $x_2^3x_3^{15}x_4^5$ | 82. $x_2^7x_3x_4^{15}$ | 83. $x_2^7x_3^3x_4^{13}$ | 84. $x_2^7x_3^9x_4^9$ |
| 85. $x_2^7x_3^{11}x_4^5$ | 86. $x_2^7x_3^{15}x_4$ | 87. $x_2^{15}x_3x_4^7$ | 88. $x_2^{15}x_3^3x_4^5$ |
| 89. $x_2^{15}x_3^7x_4$ | 90. $x_1x_2^7x_3^{15}$ | 91. $x_1x_2^{15}x_3^4x_4$ | 92. $x_1x_2x_3^6x_4^{15}$ |
| 93. $x_1x_2x_3^{15}x_4^6$ | 94. $x_1x_2^2x_3^5x_4^{15}$ | 95. $x_1x_2^2x_3^{15}x_4^5$ | 96. $x_1x_2^3x_3^4x_4^{15}$ |
| 97. $x_1x_2^3x_3^{15}x_4^4$ | 98. $x_1x_2^6x_3x_4^{15}$ | 99. $x_1x_2^6x_3^{15}x_4$ | 100. $x_1x_2^7x_4^{15}$ |
| 101. $x_1x_2^7x_3^{15}$ | 102. $x_1x_2^{15}x_3^4x_4^7$ | 103. $x_1x_2^{15}x_3^6x_4^6$ | 104. $x_1x_2^{15}x_3^2x_4^5$ |
| 105. $x_1x_2^{15}x_3^3x_4^4$ | 106. $x_1x_2^{15}x_3^6x_4$ | 107. $x_1x_2^{15}x_3^7$ | 108. $x_1^3x_3^5x_4^{15}$ |
| 109. $x_1^3x_3^7x_4^{13}$ | 110. $x_1^3x_3^{13}x_4^7$ | 111. $x_1^3x_3^{15}x_4^5$ | 112. $x_1^3x_2x_3^4x_4^{15}$ |
| 113. $x_1^3x_2x_3^{15}x_4^4$ | 114. $x_1^3x_2^4x_3x_4^{15}$ | 115. $x_1^3x_2^4x_3^{15}x_4$ | 116. $x_1^3x_2^5x_4^{15}$ |
| 117. $x_1^3x_2^5x_3^{15}$ | 118. $x_1^3x_2^7x_4^{13}$ | 119. $x_1^3x_2^7x_3^{13}$ | 120. $x_1^3x_2^{13}x_4^7$ |
| 121. $x_1^3x_2^{13}x_3^7$ | 122. $x_1^3x_2^{15}x_4^5$ | 123. $x_1^3x_2^{15}x_3x_4^4$ | 124. $x_1^3x_2^{15}x_3^4x_4$ |
| 125. $x_1^3x_2^{15}x_3^5$ | 126. $x_1^7x_3x_4^{15}$ | 127. $x_1^7x_3^3x_4^{13}$ | 128. $x_1^7x_3^7x_4^9$ |
| 129. $x_1^7x_3^{11}x_4^5$ | 130. $x_1^7x_3^{15}x_4$ | 131. $x_1^7x_2x_4^{15}$ | 132. $x_1^7x_2x_3^{15}$ |
| 133. $x_1^7x_2^3x_4^{13}$ | 134. $x_1^7x_2^3x_3^{13}$ | 135. $x_1^7x_2^9x_4^9$ | 136. $x_1^7x_2^9x_3^9$ |
| 137. $x_1^7x_2^{11}x_4^5$ | 138. $x_1^7x_2^{11}x_3^5$ | 139. $x_1^7x_2^{15}x_4$ | 140. $x_1^7x_2^{15}x_3$ |
| 141. $x_1^{15}x_3x_4^7$ | 142. $x_1^{15}x_3^3x_4^5$ | 143. $x_1^{15}x_3^7x_4$ | 144. $x_1^{15}x_2x_4^7$ |
| 145. $x_1^{15}x_2x_3x_4^6$ | 146. $x_1^{15}x_2x_3^2x_4^5$ | 147. $x_1^{15}x_2x_3^3x_4^4$ | 148. $x_1^{15}x_2x_3^6x_4$ |

$$\begin{array}{lll}
149. & x_1^{15}x_2x_3^7 & 150. & x_1^{15}x_2^3x_4^5 & 151. & x_1^{15}x_2^3x_3x_4^4 & 152. & x_1^{15}x_2^3x_3^4x_4 \\
153. & x_1^{15}x_2^3x_3^5 & 154. & x_1^{15}x_2^7x_4 & 155. & x_1^{15}x_2^7x_3.
\end{array}$$

4.3.2. *The admissible monomials of degree 23 in P_5 .*

We have $B_5(23) = B_5^0(23) \cup \psi(B_5(9)) \cup \left(B_5^+(23) \cap \text{Ker}(\widetilde{Sq}_*^0)_{(5,9)} \right)$, where $B_5^0(23) = \Phi^0(B_4(23))$, $|B_5^0(35)| = 635$, $|\psi(B_5(9))| = 191$ with $\psi : P_5 \rightarrow P_5$, $\psi(x) = X_\emptyset x^2$, and

$$B_5^+(23) \cap \text{Ker}(\widetilde{Sq}_*^0)_{(5,9)} = B_5^+(3, 2, 2, 1) \cup B_5^+(3, 4, 1, 1) \cup B_5^+(3, 4, 3).$$

$B_5^+(3, 2, 2, 1)$ is the set of 290 monomials $b_t = b_{23,t}$, $1 \leq t \leq 290$:

- | | | |
|-------------------------------------|-------------------------------------|-------------------------------------|
| 1. $x_1x_2x_3x_4^6x_5^{14}$ | 2. $x_1x_2x_3x_4^{14}x_5^6$ | 3. $x_1x_2x_3^2x_4^4x_5^{15}$ |
| 4. $x_1x_2x_3^2x_4^5x_5^{14}$ | 5. $x_1x_2x_3^2x_4^6x_5^{13}$ | 6. $x_1x_2x_3^2x_4^7x_5^{12}$ |
| 7. $x_1x_2x_3^2x_4^{12}x_5^7$ | 8. $x_1x_2x_3^2x_4^{13}x_5^6$ | 9. $x_1x_2x_3^2x_4^{14}x_5^5$ |
| 10. $x_1x_2x_3^3x_4^4x_5^{15}$ | 11. $x_1x_2x_3^3x_4^5x_5^{14}$ | 12. $x_1x_2x_3^3x_4^6x_5^{12}$ |
| 13. $x_1x_2x_3^3x_4^6x_5^{13}$ | 14. $x_1x_2x_3^3x_4^7x_5^{12}$ | 15. $x_1x_2x_3^3x_4^8x_5^{11}$ |
| 16. $x_1x_2x_3^3x_4^8x_5^{10}$ | 17. $x_1x_2x_3^3x_4^{10}x_5^9$ | 18. $x_1x_2x_3^3x_4^{11}x_5^8$ |
| 19. $x_1x_2x_3^3x_4^{12}x_5^7$ | 20. $x_1x_2x_3^3x_4^{14}x_5^5$ | 21. $x_1x_2x_3^3x_4^{15}x_5^4$ |
| 22. $x_1x_2x_3^3x_4^{14}x_5^4$ | 23. $x_1x_2x_3^3x_4^{16}x_5^3$ | 24. $x_1x_2x_3^3x_4^{17}x_5^2$ |
| 25. $x_1x_2x_3^3x_4^{18}x_5^1$ | 26. $x_1x_2x_3^3x_4^{19}x_5^0$ | 27. $x_1x_2x_3^3x_4^{20}x_5^{-1}$ |
| 28. $x_1x_2x_3^4x_4^4x_5^{15}$ | 29. $x_1x_2x_3^4x_4^5x_5^{14}$ | 30. $x_1x_2x_3^4x_4^6x_5^{13}$ |
| 31. $x_1x_2x_3^4x_4^7x_5^{12}$ | 32. $x_1x_2x_3^4x_4^{10}x_5^9$ | 33. $x_1x_2x_3^4x_4^{13}x_5^6$ |
| 34. $x_1x_2x_3^4x_4^{14}x_5^5$ | 35. $x_1x_2x_3^4x_4^{17}x_5^2$ | 36. $x_1x_2x_3^4x_4^{20}x_5^{-1}$ |
| 37. $x_1x_2x_3^4x_4^{19}x_5^0$ | 38. $x_1x_2x_3^4x_4^{21}x_5^{-1}$ | 39. $x_1x_2x_3^4x_4^{22}x_5^{-2}$ |
| 40. $x_1x_2x_3^4x_4^{23}x_5^{-3}$ | 41. $x_1x_2x_3^4x_4^{24}x_5^{-4}$ | 42. $x_1x_2x_3^4x_4^{25}x_5^{-5}$ |
| 43. $x_1x_2x_3^4x_4^{26}x_5^{-6}$ | 44. $x_1x_2x_3^4x_4^{27}x_5^{-7}$ | 45. $x_1x_2x_3^4x_4^{28}x_5^{-8}$ |
| 46. $x_1x_2x_3^4x_4^{29}x_5^{-9}$ | 47. $x_1x_2x_3^4x_4^{30}x_5^{-10}$ | 48. $x_1x_2x_3^4x_4^{31}x_5^{-11}$ |
| 49. $x_1x_2x_3^4x_4^{32}x_5^{-12}$ | 50. $x_1x_2x_3^4x_4^{33}x_5^{-13}$ | 51. $x_1x_2x_3^4x_4^{34}x_5^{-14}$ |
| 52. $x_1x_2x_3^4x_4^{35}x_5^{-15}$ | 53. $x_1x_2x_3^4x_4^{36}x_5^{-16}$ | 54. $x_1x_2x_3^4x_4^{37}x_5^{-17}$ |
| 55. $x_1x_2x_3^4x_4^{38}x_5^{-18}$ | 56. $x_1x_2x_3^4x_4^{39}x_5^{-19}$ | 57. $x_1x_2x_3^4x_4^{40}x_5^{-20}$ |
| 58. $x_1x_2x_3^4x_4^{41}x_5^{-21}$ | 59. $x_1x_2x_3^4x_4^{42}x_5^{-22}$ | 60. $x_1x_2x_3^4x_4^{43}x_5^{-23}$ |
| 61. $x_1x_2x_3^4x_4^{44}x_5^{-24}$ | 62. $x_1x_2x_3^4x_4^{45}x_5^{-25}$ | 63. $x_1x_2x_3^4x_4^{46}x_5^{-26}$ |
| 64. $x_1x_2x_3^4x_4^{47}x_5^{-27}$ | 65. $x_1x_2x_3^4x_4^{48}x_5^{-28}$ | 66. $x_1x_2x_3^4x_4^{49}x_5^{-29}$ |
| 67. $x_1x_2x_3^4x_4^{50}x_5^{-30}$ | 68. $x_1x_2x_3^4x_4^{51}x_5^{-31}$ | 69. $x_1x_2x_3^4x_4^{52}x_5^{-32}$ |
| 70. $x_1x_2x_3^4x_4^{53}x_5^{-33}$ | 71. $x_1x_2x_3^4x_4^{54}x_5^{-34}$ | 72. $x_1x_2x_3^4x_4^{55}x_5^{-35}$ |
| 73. $x_1x_2x_3^4x_4^{56}x_5^{-36}$ | 74. $x_1x_2x_3^4x_4^{57}x_5^{-37}$ | 75. $x_1x_2x_3^4x_4^{58}x_5^{-38}$ |
| 76. $x_1x_2x_3^4x_4^{59}x_5^{-39}$ | 77. $x_1x_2x_3^4x_4^{60}x_5^{-40}$ | 78. $x_1x_2x_3^4x_4^{61}x_5^{-41}$ |
| 79. $x_1x_2x_3^4x_4^{62}x_5^{-42}$ | 80. $x_1x_2x_3^4x_4^{63}x_5^{-43}$ | 81. $x_1x_2x_3^4x_4^{64}x_5^{-44}$ |
| 82. $x_1x_2x_3^4x_4^{65}x_5^{-45}$ | 83. $x_1x_2x_3^4x_4^{66}x_5^{-46}$ | 84. $x_1x_2x_3^4x_4^{67}x_5^{-47}$ |
| 85. $x_1x_2x_3^4x_4^{68}x_5^{-48}$ | 86. $x_1x_2x_3^4x_4^{69}x_5^{-49}$ | 87. $x_1x_2x_3^4x_4^{70}x_5^{-50}$ |
| 88. $x_1x_2x_3^4x_4^{71}x_5^{-51}$ | 89. $x_1x_2x_3^4x_4^{72}x_5^{-52}$ | 90. $x_1x_2x_3^4x_4^{73}x_5^{-53}$ |
| 91. $x_1x_2x_3^4x_4^{74}x_5^{-54}$ | 92. $x_1x_2x_3^4x_4^{75}x_5^{-55}$ | 93. $x_1x_2x_3^4x_4^{76}x_5^{-56}$ |
| 94. $x_1x_2x_3^4x_4^{77}x_5^{-57}$ | 95. $x_1x_2x_3^4x_4^{78}x_5^{-58}$ | 96. $x_1x_2x_3^4x_4^{79}x_5^{-59}$ |
| 97. $x_1x_2x_3^4x_4^{80}x_5^{-60}$ | 98. $x_1x_2x_3^4x_4^{81}x_5^{-61}$ | 99. $x_1x_2x_3^4x_4^{82}x_5^{-62}$ |
| 100. $x_1x_2x_3^4x_4^{83}x_5^{-63}$ | 101. $x_1x_2x_3^4x_4^{84}x_5^{-64}$ | 102. $x_1x_2x_3^4x_4^{85}x_5^{-65}$ |
| 103. $x_1x_2x_3^4x_4^{86}x_5^{-66}$ | 104. $x_1x_2x_3^4x_4^{87}x_5^{-67}$ | 105. $x_1x_2x_3^4x_4^{88}x_5^{-68}$ |
| 106. $x_1x_2x_3^4x_4^{89}x_5^{-69}$ | 107. $x_1x_2x_3^4x_4^{90}x_5^{-70}$ | 108. $x_1x_2x_3^4x_4^{91}x_5^{-71}$ |
| 109. $x_1x_2x_3^4x_4^{92}x_5^{-72}$ | 110. $x_1x_2x_3^4x_4^{93}x_5^{-73}$ | 111. $x_1x_2x_3^4x_4^{94}x_5^{-74}$ |

- | | | | | | |
|------|--------------------------------|------|--------------------------------|------|--------------------------------|
| 112. | $x_1x_2^3x_3^{14}x_4^4x_5$ | 113. | $x_1x_2^6x_3x_4x_5^{14}$ | 114. | $x_1x_2^6x_3x_4^2x_5^{13}$ |
| 115. | $x_1x_2^6x_3x_4^3x_5^{12}$ | 116. | $x_1x_2^6x_3x_4^6x_5^9$ | 117. | $x_1x_2^6x_3x_4^7x_5^8$ |
| 118. | $x_1x_2^6x_3x_4^{10}x_5^5$ | 119. | $x_1x_2^6x_3x_4^{11}x_5^4$ | 120. | $x_1x_2^6x_3x_4^{14}x_5$ |
| 121. | $x_1x_2^6x_3^3x_4x_5^{12}$ | 122. | $x_1x_2^6x_3^3x_4^9x_5^8$ | 123. | $x_1x_2^6x_3^3x_4^5x_5^8$ |
| 124. | $x_1x_2^6x_3^3x_4^8x_5^5$ | 125. | $x_1x_2^6x_3^3x_4^9x_5^4$ | 126. | $x_1x_2^6x_3^3x_4^{12}x_5$ |
| 127. | $x_1x_2^6x_3^7x_4x_5^8$ | 128. | $x_1x_2^6x_3^7x_4^8x_5$ | 129. | $x_1x_2^6x_3^9x_4^2x_5^5$ |
| 130. | $x_1x_2^6x_3^9x_4^3x_5^4$ | 131. | $x_1x_2^6x_3^{11}x_4x_5^4$ | 132. | $x_1x_2^6x_3^{11}x_4^4x_5$ |
| 133. | $x_1x_2^6x_3^2x_4x_5^{12}$ | 134. | $x_1x_2^6x_3^6x_4^8x_5^{10}$ | 135. | $x_1x_2^6x_3^4x_4^{10}x_5^4$ |
| 136. | $x_1x_2^6x_3^2x_4^5x_5^{12}$ | 137. | $x_1x_2^6x_3^2x_4^9x_5^8$ | 138. | $x_1x_2^6x_3^2x_4^5x_5^8$ |
| 139. | $x_1x_2^6x_3^2x_4^5x_5^5$ | 140. | $x_1x_2^6x_3^2x_4^9x_5^4$ | 141. | $x_1x_2^6x_3^2x_4^{12}x_5$ |
| 142. | $x_1x_2^6x_3^3x_4^8x_5^8$ | 143. | $x_1x_2^6x_3^3x_4^8x_5^4$ | 144. | $x_1x_2^6x_3^3x_4^8x_5^8$ |
| 145. | $x_1x_2^6x_3^6x_4^8x_5$ | 146. | $x_1x_2^6x_3^8x_4^2x_5^5$ | 147. | $x_1x_2^6x_3^8x_4^3x_5^4$ |
| 148. | $x_1x_2^6x_3^9x_4^2x_5^4$ | 149. | $x_1x_2^6x_3^{10}x_4x_5^4$ | 150. | $x_1x_2^6x_3^{10}x_4^4x_5$ |
| 151. | $x_1x_2^{14}x_3x_4x_5^6$ | 152. | $x_1x_2^{14}x_3^2x_4^5x_5^5$ | 153. | $x_1x_2^{14}x_3^3x_4^3x_5^4$ |
| 154. | $x_1x_2^{14}x_3x_4^6x_5$ | 155. | $x_1x_2^{14}x_3^3x_4^4x_5^4$ | 156. | $x_1x_2^{14}x_3^3x_4^4x_5$ |
| 157. | $x_1x_2^{15}x_3x_4^2x_5^4$ | 158. | $x_1x_2^{15}x_3^2x_4^4x_5^4$ | 159. | $x_1x_2^{15}x_3^2x_4^4x_5$ |
| 160. | $x_1^3x_2x_3x_4^4x_5^{14}$ | 161. | $x_1^3x_2x_3x_4^6x_5^{12}$ | 162. | $x_1^3x_2x_3x_4^{12}x_5^6$ |
| 163. | $x_1^3x_2x_3x_4^{14}x_5^4$ | 164. | $x_1^3x_2x_3^2x_4^4x_5^{13}$ | 165. | $x_1^3x_2x_3^2x_4^5x_5^{12}$ |
| 166. | $x_1^3x_2x_3^2x_4^5x_5^{12}$ | 167. | $x_1^3x_2x_3^2x_4^4x_5^{13}$ | 168. | $x_1^3x_2x_3^2x_4^4x_5^{12}$ |
| 169. | $x_1^3x_2x_3^3x_4^5x_5^4$ | 170. | $x_1^3x_2x_3^4x_4x_5^{14}$ | 171. | $x_1^3x_2x_3^4x_4^2x_5^{13}$ |
| 172. | $x_1^3x_2x_3^4x_4^5x_5^{12}$ | 173. | $x_1^3x_2x_3^4x_4^6x_5^9$ | 174. | $x_1^3x_2x_3^4x_4^7x_5^8$ |
| 175. | $x_1^3x_2x_3^4x_4^8x_5^7$ | 176. | $x_1^3x_2x_3^4x_4^9x_5^{10}$ | 177. | $x_1^3x_2x_3^4x_4^{10}x_5^5$ |
| 178. | $x_1^3x_2x_3^5x_4^{11}x_5^4$ | 179. | $x_1^3x_2x_3^4x_4^{14}x_5$ | 180. | $x_1^3x_2x_3^5x_4^2x_5^{12}$ |
| 181. | $x_1^3x_2x_3^5x_4^6x_5^8$ | 182. | $x_1^3x_2x_3^5x_4^8x_5^6$ | 183. | $x_1^3x_2x_3^5x_4^{10}x_5^4$ |
| 184. | $x_1^3x_2x_3^6x_4x_5^{12}$ | 185. | $x_1^3x_2x_3^6x_4^9x_5^8$ | 186. | $x_1^3x_2x_3^6x_4^5x_5^8$ |
| 187. | $x_1^3x_2x_3^6x_4^8x_5^5$ | 188. | $x_1^3x_2x_3^6x_4^9x_5^4$ | 189. | $x_1^3x_2x_3^6x_4^{12}x_5^6$ |
| 190. | $x_1^3x_2x_3^7x_4^8x_5^8$ | 191. | $x_1^3x_2x_3^7x_4^8x_5^4$ | 192. | $x_1^3x_2x_3^{12}x_4x_5^6$ |
| 193. | $x_1^3x_2x_3^{12}x_4^2x_5^5$ | 194. | $x_1^3x_2x_3^{12}x_4^3x_5^4$ | 195. | $x_1^3x_2x_3^{12}x_4^6x_5$ |
| 196. | $x_1^3x_2x_3^{13}x_4^2x_5^4$ | 197. | $x_1^3x_2x_3^{14}x_4x_5^4$ | 198. | $x_1^3x_2x_3^{14}x_4^4x_5$ |
| 199. | $x_1^3x_2^3x_3x_4^4x_5^{12}$ | 200. | $x_1^3x_2^3x_3^4x_4^4x_5^{12}$ | 201. | $x_1^3x_2^3x_3^4x_4^5x_5^{12}$ |
| 202. | $x_1^3x_2^3x_3^4x_4^9x_5^8$ | 203. | $x_1^3x_2^3x_3^4x_4^5x_5^8$ | 204. | $x_1^3x_2^3x_3^4x_4^5x_5^5$ |
| 205. | $x_1^3x_2^3x_3^4x_4^9x_5^4$ | 206. | $x_1^3x_2^3x_3^4x_4^{12}x_5$ | 207. | $x_1^3x_2^3x_3^5x_4^4x_5^8$ |
| 208. | $x_1^3x_2^3x_3^5x_4^8x_5^4$ | 209. | $x_1^3x_2^3x_3^{12}x_4x_5^4$ | 210. | $x_1^3x_2^3x_3^{12}x_4^4x_5$ |
| 211. | $x_1^3x_2^4x_3x_4x_5^{14}$ | 212. | $x_1^3x_2^4x_3x_4^2x_5^{13}$ | 213. | $x_1^3x_2^4x_3x_4^3x_5^{12}$ |
| 214. | $x_1^3x_2^4x_3x_4^6x_5^9$ | 215. | $x_1^3x_2^4x_3x_4^7x_5^8$ | 216. | $x_1^3x_2^4x_3x_4^{10}x_5^5$ |
| 217. | $x_1^3x_2^4x_3x_4^{11}x_5^4$ | 218. | $x_1^3x_2^4x_3x_4^{14}x_5$ | 219. | $x_1^3x_2^4x_3^3x_4^4x_5^{12}$ |
| 220. | $x_1^3x_2^4x_3^3x_4^5x_5^8$ | 221. | $x_1^3x_2^4x_3^3x_4^8x_5^5$ | 222. | $x_1^3x_2^4x_3^3x_4^9x_5^4$ |
| 223. | $x_1^3x_2^4x_3^7x_4^8x_5^8$ | 224. | $x_1^3x_2^4x_3^7x_4^8x_5^5$ | 225. | $x_1^3x_2^4x_3^9x_4^2x_5^5$ |
| 226. | $x_1^3x_2^4x_3^9x_4^3x_5^4$ | 227. | $x_1^3x_2^4x_3^{11}x_4x_5^4$ | 228. | $x_1^3x_2^5x_3x_4^2x_5^{12}$ |
| 229. | $x_1^3x_2^5x_3x_4^4x_5^8$ | 230. | $x_1^3x_2^5x_3^4x_4^{10}x_5^4$ | 231. | $x_1^3x_2^5x_3^2x_4^4x_5^{12}$ |
| 232. | $x_1^3x_2^5x_3^2x_4^9x_5^8$ | 233. | $x_1^3x_2^5x_3^2x_4^5x_5^8$ | 234. | $x_1^3x_2^5x_3^2x_4^8x_5^5$ |
| 235. | $x_1^3x_2^5x_3^2x_4^9x_5^4$ | 236. | $x_1^3x_2^5x_3^2x_4^{12}x_5$ | 237. | $x_1^3x_2^5x_3^3x_4^4x_5^8$ |
| 238. | $x_1^3x_2^5x_3^3x_4^8x_5^4$ | 239. | $x_1^3x_2^5x_3^6x_4^8x_5^8$ | 240. | $x_1^3x_2^5x_3^6x_4^8x_5^5$ |
| 241. | $x_1^3x_2^5x_3^8x_4^2x_5^5$ | 242. | $x_1^3x_2^5x_3^8x_4^3x_5^4$ | 243. | $x_1^3x_2^5x_3^9x_4^2x_5^4$ |
| 244. | $x_1^3x_2^5x_3^{10}x_4^4x_5^8$ | 245. | $x_1^3x_2^5x_3^{10}x_4^4x_5^5$ | 246. | $x_1^3x_2^7x_3x_4^4x_5^8$ |
| 247. | $x_1^3x_2^7x_3x_4^8x_5^4$ | 248. | $x_1^3x_2^7x_3^4x_4^8x_5^8$ | 249. | $x_1^3x_2^7x_3^4x_4^8x_5^5$ |
| 250. | $x_1^3x_2^8x_3x_4^8x_5^4$ | 251. | $x_1^3x_2^8x_3^8x_4^4x_5^5$ | 252. | $x_1^3x_2^{12}x_3x_4^2x_5^5$ |
| 253. | $x_1^3x_2^{12}x_3x_4^4x_5^4$ | 254. | $x_1^3x_2^{12}x_3^3x_4^4x_5^4$ | 255. | $x_1^3x_2^{13}x_3x_4^2x_5^4$ |
| 256. | $x_1^3x_2^{13}x_3^2x_4^4x_5^4$ | 257. | $x_1^3x_2^{13}x_3^2x_4^4x_5^5$ | 258. | $x_1^7x_2x_3x_4^2x_5^{12}$ |

259.	$x_1^7 x_2 x_3 x_4^6 x_5^8$	260.	$x_1^7 x_2 x_3 x_4^{10} x_5^4$	261.	$x_1^7 x_2 x_3^2 x_4 x_5^{12}$
262.	$x_1^7 x_2 x_3^2 x_4^4 x_5^9$	263.	$x_1^7 x_2 x_3^2 x_4^5 x_5^8$	264.	$x_1^7 x_2 x_3^2 x_4^8 x_5^5$
265.	$x_1^7 x_2 x_3^2 x_4^9 x_5^4$	266.	$x_1^7 x_2 x_3^2 x_4^{12} x_5$	267.	$x_1^7 x_2 x_3^3 x_4^4 x_5^8$
268.	$x_1^7 x_2 x_3^3 x_4^8 x_5^4$	269.	$x_1^7 x_2 x_3^3 x_4^8 x_5^5$	270.	$x_1^7 x_2 x_3^3 x_4^6 x_5$
271.	$x_1^7 x_2 x_3^3 x_4^5 x_5^5$	272.	$x_1^7 x_2 x_3^3 x_4^3 x_5^4$	273.	$x_1^7 x_2 x_3^3 x_4^2 x_5^4$
274.	$x_1^7 x_2 x_3^{10} x_4 x_5^4$	275.	$x_1^7 x_2 x_3^{10} x_4 x_5$	276.	$x_1^7 x_3^3 x_3^4 x_4^8 x_5$
277.	$x_1^7 x_2^3 x_3 x_4^8 x_5^4$	278.	$x_1^7 x_2^3 x_3^4 x_4^8 x_5$	279.	$x_1^7 x_2^3 x_3^4 x_4^8 x_5$
280.	$x_1^7 x_2^3 x_3^8 x_4 x_5^4$	281.	$x_1^7 x_2^3 x_3^8 x_4 x_5$	282.	$x_1^7 x_2^8 x_3 x_4^2 x_5^5$
283.	$x_1^7 x_2^8 x_3 x_4 x_5^4$	284.	$x_1^7 x_2^8 x_3 x_4 x_5$	285.	$x_1^7 x_2^9 x_3 x_4^2 x_5^4$
286.	$x_1^7 x_2^9 x_3 x_4 x_5$	287.	$x_1^7 x_2^9 x_3 x_4 x_5$	288.	$x_1^{15} x_2 x_3 x_4^2 x_5^4$
289.	$x_1^{15} x_2 x_3^2 x_4 x_5^4$	290.	$x_1^{15} x_2 x_3^2 x_4 x_5$		

$B_5^+(3, 4, 1, 1)$ is the set of 105 monomials $b_t = b_{23,t}$, $291 \leq t \leq 395$:

291.	$x_1 x_2^2 x_3^2 x_4^3 x_5^{15}$	292.	$x_1 x_2^2 x_3^2 x_4^7 x_5^{11}$	293.	$x_1 x_2^2 x_3^2 x_4^{15} x_5^3$
294.	$x_1 x_2^2 x_3^3 x_4^2 x_5^{15}$	295.	$x_1 x_2^2 x_3^3 x_4^3 x_5^{14}$	296.	$x_1 x_2^2 x_3^3 x_4^6 x_5^{11}$
297.	$x_1 x_2^2 x_3^3 x_4^7 x_5^{10}$	298.	$x_1 x_2^2 x_3^3 x_4^4 x_5^3$	299.	$x_1 x_2^2 x_3^3 x_4^{15} x_5^2$
300.	$x_1 x_2^2 x_3^7 x_4^2 x_5^{11}$	301.	$x_1 x_2^2 x_3^7 x_4^3 x_5^{10}$	302.	$x_1 x_2^2 x_3^7 x_4^4 x_5^3$
303.	$x_1 x_2^2 x_3^7 x_4^{11} x_5^2$	304.	$x_1 x_2^2 x_3^{15} x_4^2 x_5^3$	305.	$x_1 x_2^2 x_3^{15} x_4^3 x_5^2$
306.	$x_1 x_2^2 x_3^2 x_4^2 x_5^{15}$	307.	$x_1 x_2^2 x_3^2 x_4^3 x_5^{14}$	308.	$x_1 x_2^2 x_3^2 x_4^6 x_5^{11}$
309.	$x_1 x_2^2 x_3^2 x_4^7 x_5^{10}$	310.	$x_1 x_2^2 x_3^2 x_4^{14} x_5^3$	311.	$x_1 x_2^2 x_3^2 x_4^{15} x_5^2$
312.	$x_1 x_2^2 x_3^3 x_4^2 x_5^{14}$	313.	$x_1 x_2^2 x_3^3 x_4^6 x_5^{10}$	314.	$x_1 x_2^2 x_3^3 x_4^{14} x_5^2$
315.	$x_1 x_2^2 x_3^6 x_4^2 x_5^{11}$	316.	$x_1 x_2^2 x_3^6 x_4^3 x_5^{10}$	317.	$x_1 x_2^2 x_3^6 x_4^{10} x_5^3$
318.	$x_1 x_2^2 x_3^6 x_4^{11} x_5^2$	319.	$x_1 x_2^2 x_3^7 x_4^2 x_5^{10}$	320.	$x_1 x_2^2 x_3^7 x_4^{10} x_5^2$
321.	$x_1 x_2^2 x_3^{14} x_4^2 x_5^3$	322.	$x_1 x_2^2 x_3^{14} x_4^3 x_5^2$	323.	$x_1 x_2^2 x_3^{15} x_4^2 x_5^2$
324.	$x_1 x_2^2 x_3^2 x_4^2 x_5^{11}$	325.	$x_1 x_2^2 x_3^2 x_4^3 x_5^{10}$	326.	$x_1 x_2^2 x_3^2 x_4^{10} x_5^3$
327.	$x_1 x_2^2 x_3^2 x_4^{11} x_5^2$	328.	$x_1 x_2^2 x_3^2 x_4^{10} x_5^2$	329.	$x_1 x_2^2 x_3^3 x_4^{10} x_5^2$
330.	$x_1 x_2^2 x_3^{10} x_4^2 x_5^3$	331.	$x_1 x_2^2 x_3^{10} x_4^3 x_5^2$	332.	$x_1 x_2^2 x_3^{11} x_4^2 x_5^2$
333.	$x_1 x_2^{15} x_3^2 x_4^2 x_5^3$	334.	$x_1 x_2^{15} x_3^2 x_4^3 x_5^2$	335.	$x_1 x_2^{15} x_3^3 x_4^2 x_5^2$
336.	$x_1^3 x_2^2 x_3^2 x_4^2 x_5^{15}$	337.	$x_1^3 x_2^2 x_3^2 x_4^3 x_5^{14}$	338.	$x_1^3 x_2^2 x_3^2 x_4^6 x_5^{11}$
339.	$x_1^3 x_2^2 x_3^2 x_4^7 x_5^{10}$	340.	$x_1^3 x_2^2 x_3^2 x_4^{14} x_5^3$	341.	$x_1^3 x_2^2 x_3^2 x_4^{15} x_5^2$
342.	$x_1^3 x_2^2 x_3^3 x_4^2 x_5^{14}$	343.	$x_1^3 x_2^2 x_3^3 x_4^6 x_5^{10}$	344.	$x_1^3 x_2^2 x_3^3 x_4^{14} x_5^2$
345.	$x_1^3 x_2^2 x_3^6 x_4^2 x_5^{11}$	346.	$x_1^3 x_2^2 x_3^6 x_4^3 x_5^{10}$	347.	$x_1^3 x_2^2 x_3^6 x_4^{10} x_5^3$
348.	$x_1^3 x_2^2 x_3^6 x_4^{11} x_5^2$	349.	$x_1^3 x_2^2 x_3^7 x_4^2 x_5^{10}$	350.	$x_1^3 x_2^2 x_3^7 x_4^{10} x_5^2$
351.	$x_1^3 x_2^2 x_3^{14} x_4^2 x_5^3$	352.	$x_1^3 x_2^2 x_3^{14} x_4^3 x_5^2$	353.	$x_1^3 x_2^2 x_3^{15} x_4^2 x_5^2$
354.	$x_1^3 x_2^3 x_3^2 x_4^2 x_5^{14}$	355.	$x_1^3 x_2^3 x_3^2 x_4^3 x_5^{10}$	356.	$x_1^3 x_2^3 x_3^2 x_4^{14} x_5^2$
357.	$x_1^3 x_2^3 x_3^2 x_4^7 x_5^{10}$	358.	$x_1^3 x_2^3 x_3^2 x_4^{10} x_5^2$	359.	$x_1^3 x_2^3 x_3^3 x_4^2 x_5^2$
360.	$x_1^3 x_2^3 x_3^5 x_4^2 x_5^{11}$	361.	$x_1^3 x_2^3 x_3^5 x_4^3 x_5^{10}$	362.	$x_1^3 x_2^3 x_3^5 x_4^{10} x_5^3$
363.	$x_1^3 x_2^3 x_3^5 x_4^{11} x_5^2$	364.	$x_1^3 x_2^3 x_3^5 x_4^{10} x_5^2$	365.	$x_1^3 x_2^3 x_3^5 x_4^{10} x_5^2$
366.	$x_1^3 x_2^3 x_3^{10} x_4^2 x_5^3$	367.	$x_1^3 x_2^3 x_3^{10} x_4^3 x_5^2$	368.	$x_1^3 x_2^3 x_3^{11} x_4^2 x_5^2$
369.	$x_1^3 x_2^3 x_3^{10} x_4^{10} x_5^2$	370.	$x_1^3 x_2^3 x_3^{10} x_4^{10} x_5^2$	371.	$x_1^3 x_2^3 x_3^9 x_4^2 x_5^2$
372.	$x_1^3 x_2^{13} x_3^2 x_4^2 x_5^3$	373.	$x_1^3 x_2^{13} x_3^2 x_4^3 x_5^2$	374.	$x_1^3 x_2^{13} x_3^3 x_4^2 x_5^2$
375.	$x_1^3 x_2^{15} x_3^2 x_4^2 x_5^2$	376.	$x_1^7 x_2 x_3^2 x_4^2 x_5^{11}$	377.	$x_1^7 x_2 x_3^2 x_4^3 x_5^{10}$
378.	$x_1^7 x_2 x_3^2 x_4^{10} x_5^3$	379.	$x_1^7 x_2 x_3^2 x_4^{11} x_5^2$	380.	$x_1^7 x_2 x_3^2 x_4^{10} x_5^2$
381.	$x_1^7 x_2 x_3^3 x_4^{10} x_5^2$	382.	$x_1^7 x_2 x_3^{10} x_4^2 x_5^3$	383.	$x_1^7 x_2 x_3^{10} x_4^3 x_5^2$
384.	$x_1^7 x_2 x_3^{11} x_4^2 x_5^2$	385.	$x_1^7 x_2 x_3^{10} x_4^2 x_5^{10}$	386.	$x_1^7 x_2 x_3^{10} x_4^{10} x_5^2$
387.	$x_1^7 x_2^9 x_3^2 x_4^2 x_5^2$	388.	$x_1^7 x_2^9 x_3^2 x_4^3 x_5^2$	389.	$x_1^7 x_2^9 x_3^2 x_4^3 x_5^2$
390.	$x_1^7 x_2^9 x_3^2 x_4^2 x_5^2$	391.	$x_1^7 x_2^{11} x_3^2 x_4^2 x_5^2$	392.	$x_1^{15} x_2 x_3^2 x_4^2 x_5^3$
393.	$x_1^{15} x_2 x_3^2 x_4^2 x_5^2$	394.	$x_1^{15} x_2 x_3^2 x_4^2 x_5^2$	395.	$x_1^{15} x_2 x_3^2 x_4^2 x_5^2$

$B_5^+(3, 4, 3)$ is the set of 24 monomials $b_t = b_{23,t}$, $396 \leq t \leq 419$:

$$\begin{array}{llll}
396. & x_1 x_2^3 x_3^6 x_4^6 x_5^7 & 397. & x_1 x_2^3 x_3^6 x_4^7 x_5^6 \\
400. & x_1^3 x_2 x_3^6 x_4^6 x_5^7 & 401. & x_1^3 x_2 x_3^6 x_4^7 x_5^6 \\
404. & x_1^3 x_2^5 x_3^6 x_4^6 x_5^7 & 405. & x_1^3 x_2^5 x_3^6 x_4^7 x_5^6 \\
408. & x_1^3 x_2^5 x_3^6 x_4^6 x_5^7 & 409. & x_1^3 x_2^5 x_3^6 x_4^7 x_5^6 \\
412. & x_1^3 x_2^5 x_3^6 x_4^6 x_5^7 & 413. & x_1^3 x_2^5 x_3^6 x_4^7 x_5^6 \\
416. & x_1^7 x_2 x_3^6 x_4^6 x_5^6 & 417. & x_1^7 x_2 x_3^6 x_4^7 x_5^6 \\
& & 418. & x_1^7 x_2 x_3^6 x_4^6 x_5^6 \\
& & 419. & x_1^7 x_2 x_3^6 x_4^7 x_5^6
\end{array}$$

4.4. Some Σ_5 -invariant classes of degree 23 in P_5 .

We list here some polynomials which present the Σ_5 -invariant classes of degree 23 in QP_5 .

$$\begin{aligned}
p_4 = & x_1 x_2^2 x_3^3 x_4^3 x_5^{14} + x_1 x_2^2 x_3^3 x_4^{14} x_5^3 + x_1 x_2^2 x_3^7 x_4^3 x_5^{10} + x_1 x_2^2 x_3^7 x_4^{10} x_5^3 \\
& + x_1 x_2^3 x_3^2 x_4^3 x_5^{14} + x_1 x_2^3 x_3^2 x_4^{14} x_5^3 + x_1 x_2^3 x_3^2 x_4^3 x_5^{14} + x_1 x_2^3 x_3^2 x_4^{14} x_5^3 \\
& + x_1 x_2^3 x_3^{14} x_4^2 x_5^3 + x_1 x_2^3 x_3^{14} x_4^3 x_5^2 + x_1 x_2^7 x_3^2 x_4^3 x_5^{10} + x_1 x_2^7 x_3^2 x_4^{10} x_5^3 \\
& + x_1 x_2^7 x_3^2 x_4^3 x_5^{10} + x_1 x_2^7 x_3^2 x_4^{10} x_5^3 + x_1 x_2^7 x_3^{10} x_4^2 x_5^3 + x_1 x_2^7 x_3^{10} x_4^3 x_5^2 \\
& + x_1^3 x_2 x_3^2 x_4^3 x_5^{14} + x_1^3 x_2 x_3^2 x_4^{14} x_5^3 + x_1^3 x_2 x_3^2 x_4^3 x_5^{14} + x_1^3 x_2 x_3^2 x_4^{14} x_5^3 \\
& + x_1^3 x_2 x_3^{14} x_4^2 x_5^3 + x_1^3 x_2 x_3^{14} x_4^3 x_5^2 + x_1^3 x_2 x_3^{14} x_4^2 x_5^3 + x_1^3 x_2 x_3^{14} x_4^3 x_5^2 \\
& + x_1^3 x_2 x_3^{13} x_4^2 x_5^2 + x_1^3 x_2 x_3^{13} x_4^3 x_5^2 + x_1^3 x_2 x_3^{13} x_4^2 x_5^2 + x_1^3 x_2 x_3^{13} x_4^3 x_5^2 \\
& + x_1^7 x_2 x_3^2 x_4^3 x_5^{10} + x_1^7 x_2 x_3^2 x_4^{10} x_5^3 + x_1^7 x_2 x_3^2 x_4^3 x_5^{10} + x_1^7 x_2 x_3^2 x_4^{10} x_5^3 \\
& + x_1^7 x_2 x_3^{10} x_4^2 x_5^3 + x_1^7 x_2 x_3^{10} x_4^3 x_5^2 + x_1^7 x_2 x_3^{10} x_4^2 x_5^3 + x_1^7 x_2 x_3^{10} x_4^3 x_5^2 \\
& + x_1^7 x_2 x_3^9 x_4^2 x_5^2 + x_1^7 x_2 x_3^9 x_4^3 x_5^2 + x_1^7 x_2 x_3^9 x_4^2 x_5^2 + x_1^7 x_2 x_3^9 x_4^3 x_5^2.
\end{aligned}$$

$$\begin{aligned}
p_5 = & x_2 x_3 x_4^7 x_5^{14} + x_2 x_3^7 x_4 x_5^{14} + x_2 x_3^7 x_4^{14} x_5 + x_2^7 x_3 x_4 x_5^{14} + x_2^7 x_3 x_4^{14} x_5 \\
& + x_1 x_3 x_4^7 x_5^{14} + x_1 x_3^7 x_4 x_5^{14} + x_1 x_3^7 x_4^{14} x_5 + x_1 x_2 x_4^7 x_5^{14} + x_1 x_2 x_3^7 x_5^{14} \\
& + x_1 x_2 x_3^7 x_4^{14} + x_1 x_2^7 x_4 x_5^{14} + x_1 x_2^7 x_4^{14} x_5 + x_1 x_2^7 x_3 x_5^{14} + x_1 x_2^7 x_3 x_4^{14} \\
& + x_1 x_2^7 x_3^{14} x_5 + x_1 x_2^7 x_3^{14} x_4 + x_1^7 x_3 x_4 x_5^{14} + x_1^7 x_3 x_4^{14} x_5 + x_1^7 x_2 x_4 x_5^{14} \\
& + x_1^7 x_2 x_4^{14} x_5 + x_1^7 x_2 x_3 x_5^{14} + x_1^7 x_2 x_3 x_4^{14} + x_1^7 x_2 x_3^{14} x_5 + x_1^7 x_2 x_3^{14} x_4 \\
& + x_2 x_3^3 x_4^{13} x_5^6 + x_2 x_3^3 x_4^{13} x_5^6 + x_2 x_3^3 x_4^{13} x_5^6 + x_2 x_3^3 x_4^{13} x_5^6 + x_1 x_3^3 x_4^{13} x_5^6 \\
& + x_1 x_2^3 x_4^{13} x_5^6 + x_1 x_2^3 x_3^{13} x_5^6 + x_1 x_2^3 x_3^{13} x_4^6 + x_1^3 x_3 x_4^{13} x_5^6 + x_1^3 x_3^{13} x_4 x_5^6 \\
& + x_1^3 x_3^{13} x_4^6 x_5 + x_1^3 x_2 x_4^{13} x_5^6 + x_1^3 x_2 x_3^{13} x_5^6 + x_1^3 x_2 x_3^{13} x_4^6 + x_1^3 x_2^{13} x_4 x_5^6 \\
& + x_1^3 x_2^{13} x_4^6 x_5 + x_1^3 x_2^{13} x_3 x_5^6 + x_1^3 x_2^{13} x_3 x_4^6 + x_1^3 x_2^{13} x_3^6 x_5 + x_1^3 x_2^{13} x_3^6 x_4 \\
& + x_2^7 x_3^{11} x_4 x_5^4 + x_2^7 x_3^{11} x_4^4 x_5 + x_1^7 x_3^{11} x_4 x_5^4 + x_1^7 x_3^{11} x_4^4 x_5 + x_1^7 x_2^{11} x_4 x_5^4 \\
& + x_1^7 x_2^{11} x_4^4 x_5 + x_1^7 x_2^{11} x_3 x_5^4 + x_1^7 x_2^{11} x_3 x_4^4 + x_1^7 x_2^{11} x_3^4 x_5 + x_1^7 x_2^{11} x_3^4 x_4 \\
& + x_2 x_3^6 x_4^7 x_5^9 + x_1 x_3^6 x_4^7 x_5^9 + x_1 x_2^6 x_4^7 x_5^9 + x_1 x_2^6 x_3^7 x_5^9 + x_1 x_2^6 x_3^7 x_4^9 \\
& + x_2^3 x_3^4 x_4^7 x_5^9 + x_1^3 x_3^4 x_4^7 x_5^9 + x_1^3 x_2^4 x_4^7 x_5^9 + x_1^3 x_2^4 x_3^7 x_5^9 + x_1^3 x_2^4 x_3^7 x_4^9.
\end{aligned}$$

$$\begin{aligned}
p_6 = & x_2x_3^7x_4^8x_5^8 + x_2^7x_3x_4^7x_5^8 + x_2^7x_3^7x_4x_5^8 + x_2^7x_3^7x_4^8x_5 + x_1x_3^7x_4^7x_5^8 \\
& + x_1x_2^7x_4^7x_5^8 + x_1x_2^7x_3^7x_5^8 + x_1x_2^7x_3^7x_4^8 + x_1^7x_3x_4^7x_5^8 + x_1^7x_3^7x_4x_5^8 \\
& + x_1^7x_3^7x_4^8x_5 + x_1^7x_2x_4^7x_5^8 + x_1^7x_2x_3^7x_5^8 + x_1^7x_2x_3^7x_4^8 + x_1^7x_2^7x_4x_5^8 \\
& + x_1^7x_2^7x_4^8x_5 + x_1^7x_2^7x_3x_5^8 + x_1^7x_2^7x_3x_4^8 + x_1^7x_2^7x_3^8x_5 + x_1^7x_2^7x_3^8x_4 \\
& + x_2x_3^3x_4^7x_5^{12} + x_2^7x_3^3x_4x_5^{12} + x_2^7x_3^3x_4^12x_5 + x_1x_3^3x_4^7x_5^{12} + x_1x_2^3x_4^7x_5^{12} \\
& + x_1x_2^3x_3^7x_5^{12} + x_1x_2^3x_3^7x_4^{12} + x_1^7x_3^3x_4x_5^{12} + x_1^7x_3^3x_4^12x_5 + x_1^7x_2^3x_4x_5^{12} \\
& + x_1^7x_2^3x_4^12x_5 + x_1^7x_2^3x_3x_5^{12} + x_1^7x_2^3x_3x_4^{12} + x_1^7x_2^3x_3^12x_5 + x_1^7x_2^3x_3^12x_4 \\
& + x_2x_3^7x_4^6x_5^9 + x_2^7x_3x_4^6x_5^9 + x_1x_3^7x_4^6x_5^9 + x_1x_2^7x_4^6x_5^9 + x_1x_2^7x_3^6x_5^9 \\
& + x_1x_2^7x_3^6x_4^9 + x_1^7x_3x_4^6x_5^9 + x_1^7x_2x_4^6x_5^9 + x_1^7x_2x_3^6x_5^9 + x_1^7x_2x_3^6x_4^9 \\
& + x_2^3x_3^7x_4^9x_5^4 + x_2^3x_3^7x_4^9x_5^4 + x_2^7x_3^3x_4^9x_5^4 + x_2^7x_3^3x_4^9x_5^4 + x_1^3x_3^7x_4^9x_5^4 \\
& + x_1^3x_3^7x_4^9x_5^4 + x_1^3x_2^7x_4^9x_5^4 + x_1^3x_2^7x_4^9x_5^4 + x_1^3x_2^7x_3^9x_5^4 + x_1^3x_2^7x_3^9x_4^9 \\
& + x_1^3x_2^7x_3^9x_5^4 + x_1^3x_2^7x_3^9x_4^9 + x_1^7x_3^3x_4^9x_5^4 + x_1^7x_3^3x_4^9x_5^4 + x_1^7x_2^3x_4^9x_5^4 \\
& + x_1^7x_2^3x_4^9x_5^4 + x_1^7x_2^3x_3^9x_5^4 + x_1^7x_2^3x_3^9x_4^9 + x_1^7x_2^3x_3^9x_4^9 + x_1^7x_2^3x_3^9x_4^9.
\end{aligned}$$

$$\begin{aligned}
p_7 = & x_2x_3^3x_4^5x_5^{14} + x_2x_3^3x_4^{14}x_5^5 + x_2^3x_3x_4^5x_5^{14} + x_2^3x_3^5x_4x_5^{14} + x_2^3x_3^5x_4^{14}x_5 \\
& + x_1x_3^3x_4^5x_5^{14} + x_1x_3^3x_4^{14}x_5^5 + x_1x_2^3x_4^5x_5^{14} + x_1x_2^3x_4^{14}x_5^5 + x_1x_2^3x_3^5x_5^{14} \\
& + x_1x_2^3x_3^5x_4^{14} + x_1x_2^3x_3^{14}x_5^5 + x_1x_2^3x_3^{14}x_4^5 + x_1^3x_3x_4^5x_5^{14} + x_1^3x_3^5x_4x_5^{14} \\
& + x_1^3x_3^5x_4^{14}x_5 + x_1^3x_2x_4^5x_5^{14} + x_1^3x_2x_3^5x_5^{14} + x_1^3x_2x_3^5x_4^{14} + x_1^3x_2^5x_4x_5^{14} \\
& + x_1^3x_2^5x_4^{14}x_5 + x_1^3x_2^5x_3x_5^{14} + x_1^3x_2^5x_3x_4^{14} + x_1^3x_2^5x_3^{14}x_5 + x_1^3x_2^5x_3^{14}x_4 \\
& + x_2x_3^3x_4^6x_5^{13} + x_2x_3^3x_4^6x_5^{13} + x_1x_3^3x_4^6x_5^{13} + x_1x_3^3x_4^6x_5^{13} + x_1x_2^3x_4^6x_5^{13} \\
& + x_1x_2^3x_3^6x_5^{13} + x_1x_2^3x_3^6x_4^{13} + x_1x_2^6x_3^6x_5^{13} + x_1x_2^6x_3^6x_5^{13} + x_1x_2^6x_3^6x_4^{13} \\
& + x_2^3x_3x_4^7x_5^{12} + x_2^3x_3x_4^7x_5^{12} + x_2^3x_3^7x_4x_5^{12} + x_2^3x_3^7x_4x_5^{12} + x_1^3x_3x_4^7x_5^{12} \\
& + x_1^3x_3x_4^7x_5^{12} + x_1^3x_2x_4^7x_5^{12} + x_1^3x_2x_3^7x_5^{12} + x_1^3x_2x_3^7x_4^{12} + x_1^3x_2x_3^7x_4^{12} \\
& + x_1^3x_2x_3^7x_4^{12}x_5 + x_1^3x_2x_3^7x_4^{12}x_5 + x_1^3x_2x_3^7x_4^{12}x_5 + x_1^3x_2x_3^7x_4^{12}x_5 + x_1^3x_2x_3^7x_4^{12}x_5 \\
& + x_2x_3^6x_4^{11}x_5^5 + x_1x_3^6x_4^{11}x_5^5 + x_1x_2^6x_4^{11}x_5^5 + x_1x_2^6x_3^{11}x_5^5 + x_1x_2^6x_3^{11}x_4^5 \\
& + x_2^3x_3^4x_4^{13} + x_2^3x_3^4x_4^{13}x_5 + x_2^3x_3^4x_4^{13}x_5 + x_2^3x_3^4x_4^{13}x_5 + x_2^3x_3^4x_4^{13}x_5 \\
& + x_1^3x_3^4x_4^{13} + x_1^3x_3^4x_4^{13}x_5 + x_1^3x_3^4x_4^{13}x_5 + x_1^3x_3^4x_4^{13}x_5 + x_1^3x_3^4x_4^{13}x_5 \\
& + x_1^3x_2^3x_3^{13}x_5^4 + x_1^3x_2^3x_3^{13}x_4^4 + x_1^3x_2^3x_3^{13}x_4^4 + x_1^3x_2^3x_3^{13}x_4^4 + x_1^3x_2^3x_3^{13}x_4^4 \\
& + x_2^3x_3^4x_4^{11}x_5^5 + x_1x_3^4x_4^{11}x_5^5 + x_1x_2^3x_4^{11}x_5^5 + x_1x_2^3x_4^{11}x_5^5 + x_1x_2^3x_4^{11}x_5^5 \\
& + x_2^3x_3^3x_4^5x_5^{12} + x_2^3x_3^3x_4^5x_5^{12} + x_1^3x_3^3x_4^5x_5^{12} + x_1^3x_3^3x_4^5x_5^{12} + x_1^3x_2^3x_4^5x_5^{12} \\
& + x_1^3x_2^3x_4^5x_5^{12} + x_1^3x_2^3x_3^5x_5^{12} + x_1^3x_2^3x_3^5x_4^{12} + x_1^3x_2^3x_3^5x_4^{12} + x_1^3x_2^3x_3^5x_4^{12}
\end{aligned}$$

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